DIOPHANTINE GEOMETRY OVER GROUPS IX: ENVELOPES AND IMAGINARIES

Z. Sela[1,2]

This paper is the ninth in a sequence on the structure of sets of solutions to systems of equations in free and hyperbolic groups, projections of such sets (Diophantine sets), and the structure of definable sets over free and hyperbolic groups. In the ninth paper we associate a Diophantine set with a definable set, and view it as the Diophantine envelope of the definable set. We use the envelope and duo limit groups that were used in proving stability of the theory of free and torsion-free hyperbolic groups [Se9], to study definable equivalence relations, and in particular, to classify imaginaries over these groups.

In the first 6 papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects that are required for the analysis of sentences and elementary sets defined over a free group. The techniques we developed, enabled us to present an iterative procedure that analyzes EAE sets defined over a free group (i.e., sets defined using 3 quantifiers), and shows that every such set is in the Boolean algebra generated by AE sets ([Se6],41), hence, we obtained a quantifier elimination over a free group.
In the 7th paper in the sequence we generalized the techniques and the results from free groups to torsion-free hyperbolic groups, and in the 8th paper we used the techniques that were developed for quantifier elimination to prove that the elementary theories of free and torsion-free hyperbolic groups are stable.

In the 9th paper in the sequence we study definable equivalence relations over free and hyperbolic groups. The understanding of the structure of definable equivalence relations is central in model theory (see [Pi1] and [Pi2]), and in particular it is necessary in order to study what can be interpreted in the theories of these groups.

In an arbitrary group, there are 3 basic (not necessarily definable) families of equivalence relations: conjugation, left and right cosets of subgroups, and double cosets of subgroups. As in general a subgroup may not be definable, not all these equivalence relations are definable equivalence relations.

By results of M. Bestvina and M. Feighn [Be-Fe2] (on negligible sets), the only definable subgroups of a free or a torsion-free hyperbolic group are (infinite) cyclic. Hence, the only basic equivalence relations over these groups that are definable, are conjugation, and left, right and double cosets of cyclic groups. Note that conjugation is an equivalence relation of singletons, left and right cosets of cyclic groups are equivalence relations of pairs of elements, and double cosets of cyclic

[1] Hebrew University, Jerusalem 91904, Israel.
[2] Partially supported by an Israel academy of sciences fellowship.

groups are equivalence relations of triples of elements (see theorems 2.1, 2.2 and 2.3).

Our first goal in this paper is to show that these 3 basic families of definable equivalence relations are imaginaries (i.e., not reals). In model theory, a (definable) equivalence relation is considered trivial (and called real) if it is obtained from a definable function, i.e., if there exists some definable function so that every equivalence class is the preimage of a point. In the second section of this paper (theorems 2.1-2.4) we show that over free and torsion-free hyperbolic groups, conjugation, left and right cosets of cyclic groups, and double cosets of cyclic groups are not reals (i.e., there exists no definable function so that classes in any of these equivalence relations are preimages of points).

The main goal of this paper is to classify (or represent) all the definable equivalence relations over free and torsion-free hyperbolic groups. In particular, we aim at classifying all the imaginaries (non-reals) over these groups. Our concluding theorems (theorems 4.4 and 4.6) show that the basic definable equivalence relations are the only "essential" imaginaries. In particular, we show that if sorts are added for the 3 basic families of imaginaries (conjugation, left, right and double cosets of cyclic groups), then (definable) equivalence relations can be geometrically eliminated. Geometric elimination means that if G is a free or torsion-free hyperbolic group, p and q are m-tuples, and $E(p,q)$, is a definable equivalence relation, then there exist some integers s and t, and a definable multi-function:

$$f : G^m \ \to \ G^s \times R_1 \times \ldots \times R_t$$

where each of the R_i's is a new sort for one of the 3 basic families of imaginaries (conjugation, left, right and double cosets of cyclic groups), the image of an element is uniformly bounded (and can be assumed to be of equal size), the multi-function is a class function, i.e., two elements in an equivalence class of $E(p,q)$ have the same image, and the multi-function f separates between classes, i.e., the images of elements from distinct equivalence classes are distinct. Furthermore, if $E(p,q)$ is coefficient-free, then we can choose the definable multi-function f to be coefficient-free.

In fact, we prove more than geometric elimination of imaginaries, as we do get a representation of (generic points in) the equivalence classes of a given definable equivalence relation as fibers over some (definable) parameters, where the definable parameters separate between classes, and for each equivalence class the definable parameters admit only boundedly many values up to the basic (definable) equivalence relations.

The main tool that we use for analyzing definable equivalence relations is the *Diophantine envelope* of a definable set. Given a definable set, $L(p)$, we associate with it a Diophantine set, $D(p)$, its Diophantine envelope. $L(p) \subset D(p)$, and for a certain notion of (combinatorial) genericity (which differs from the stability theoretic one), a generic point in the envelope, $D(p)$, is contained in the original definable set, $L(p)$. If the free variables in the definable set are divided into two tuples, $L(p,q)$, then we associate with $L(p,q)$ a *Duo* (Diophantine) *Envelope*, $Duo(p,q)$, with a dual Duo limit group (Duo limit groups were introduced in section 3 of [Se9], and served as the main tool in proving stability of free and torsion-free hyperbolic groups). Again, $L(p,q) \subset Duo(p,q)$, and (combinatorial) generic points in $Duo(p,q)$ are contained in $L(p,q)$.

The first section of the paper constructs the Diophantine and Duo envelopes of definable sets. The construction of the envelopes is based on the sieve procedure [Se6], that was originally used for quantifier elimination. The sieve procedure is the main technical tool in proving stability of the theory [Se9], equationality of Diophantine sets [Se9], and is also the main technical tool in analyzing equivalence relations in this paper.

In the second section we use the existence of the Diophantine envelope (and its properties), to prove that the 3 basic families of equivalence relations over free and torsion-free hyperbolic groups: conjugation, left and right cosets of cyclic groups, and double cosets of cyclic groups, are imaginaries (not reals). We believe that envelopes can serve as an applicable tool to prove non-definability (and at times definability) in many other cases.

In the third section we start analyzing general definable equivalence relations. The Diophantine and Duo envelopes that are constructed in the first section of the paper, depend on the defining predicate, and in particular are not canonical. Our strategy in associating parameters to equivalence classes relies on a procedure for constructing canonical envelopes.

Given a (definable) equivalence relation, $E(p,q)$, we start with its Duo envelope, and gradually modify it. Into each of the iterative sequence of envelopes that we construct, that we call $uniformization$ limit groups, there exists a map from a group that specializes to valid proofs that the specializations of the tuples, (p,q), are indeed in the given definable equivalence relation, $E(p,q)$. We use this map to associate parameters with the equivalence classes of the given equivalence relation, $E(p,q)$. The image of this map inherits a graphs of groups decomposition from the constructed (ambient) uniformization limit group, a graph of groups in which the subgroup $< p >$ is contained in one vertex group, the subgroup $< q >$ is contained in a second vertex group, and for each equivalence class of $E(p,q)$, the edge groups in the inherited graph of groups are generated by finitely many elements, where these elements admit only (uniformly) boundedly many values up to the basic equivalence relations (conjugation, left right, and double cosets of cyclic groups).

The parameters that are associated with the edge groups in the graphs of groups that are inherited from the uniformization limit groups, give us parameters with only boundedly many values for each equivalence class (up to the basic imaginaries). Hence, it is possible to construct a definable (class) multi-function using them. However, these parameters are not guaranteed to separate between classes. Hence, these parameters and graphs of groups are not sufficient for obtaining geometric elimination of imaginaries.
Still, the graphs of groups that are inherited from the constructed uniformization limit groups, and their associated parameters, enable us to separate variables, i.e., to separate the subgroup $< p >$ from the subgroup $< q >$. These subgroups are contained in two distinct vertex groups in the graphs of groups, and the edge groups are generated by finitely many elements that admit only boundedly many values (up to the basic imaginaries) for each equivalence class.

The separation of variables that is obtained by the graphs of groups that are inherited from the uniformization limit groups, is the key for obtaining geometric elimination of imaginaries in the fourth section of the paper. In this section we present another iterative procedure, that combines the sieve procedure [Se6], with the procedure for separation of variables that is presented in section 3. The combined procedure, iteratively constructs smaller and smaller (Duo) Diophantine

sets, that converge after finitely many steps to a Diophantine (Duo) envelope of the equivalence relation, $E(p,q)$. Unlike the (Duo) envelope that is constructed in the first section, the envelope that is constructed by this combined iterative procedure is canonical. This means that the envelope is determined by the value of finitely many elements, and these elements admit only (uniformly) boundedly many values for each equivalence class of $E(p,q)$ (up to the basic imaginaries). Therefore, the parameters that are associated with the envelope that is constructed by the combined procedure, can be used to define the desired multi-function, that finally proves geometric elimination of imaginaries (theorems 4.4 and 4.6).

We believe that some of the techniques, notions and constructions that appear in this paper can be used to study other model theoretic properties of free, torsion-free hyperbolic, and other groups. The arguments that we use also demonstrate the power and the applicability of the sieve procedure [Se6] for tackling model theoretic problems and properties.

Finally, we would like to thank Gregory Cherlin for suggesting the study of equivalence relations to us, and Ehud Hrushovsky and Eliyahu Rips for their constant advice.

§1. Diophantine and Duo Envelopes

Before we analyze some of the basic imaginaries over free and hyperbolic groups, we present two of the main tools that are needed in order to classify the entire collection of imaginaries over free and torsion-free hyperbolic groups, which may also serve as a tool in proving that certain sets are not definable. First, we recall the definitions of a Duo limit group, and its associated Duo families, and some of their properties (section 3 in [Se9]). Then given a definable set, we associate with it a (canonical) finite collection of graded limit groups that together form a *Diophantine envelope* of the definable set, and with them we associate a canonical collection of duo limit groups, that in certain cases can be viewed as a *Duo envelope*.

Definition 1.1 ([Se9],3.1). *Let F_k be a non-abelian free group, and let $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$) be a rigid (solid) limit group with respect to the parameter subgroup $<p,q>$. Let s be a (fixed) positive integer, and let $Conf(x_1,\dots,x_s,p,q,a)$ be a configuration limit group associated with the limit group $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$) (see definition 4.1 in [Se3] for configuration limit groups). Recall that a configuration limit group is obtained as a limit of a convergent sequence of specializations $(x_1(n),\dots,x_s(n),p(n),q(n),a)$, called configuration homomorphisms ([Se3],4.1), in which each of the specializations $(x_i(n),p(n),q(n),a)$ is rigid (strictly solid) and $x_i(n)\neq x_j(n)$ for $i\neq j$ (belong to distinct strictly solid families). See section 4 of [Se3] for a detailed discussion of these groups.*

A duo limit group, $Duo(d_1,p,d_2,q,d_0,a)$, is a limit group with the following properties:

(1) *with Duo there exists an associated map:*

$$\eta: Conf(x_1,\dots,x_s,p,q,a)\to Duo.$$

For brevity, we denote $\eta(p),\eta(q),\eta(a)$ by p,q,a in correspondence.

(2) *$Duo=<d_1>*_{<d_0,e_1>}<d_0,e_1,e_2>*_{<d_0,e_2>}<d_2>$, $\eta(F_k)=\eta(<a>)<<d_0>$, $\eta(<p>)<<d_1>$, and $\eta(<q>)<<d_2>$.*

(3) *$Duo = Comp(d_1,p,a) *_{<d_0,e_1>} < d_0,e_1,e_2 > *_{<d_0,e_2>} Comp(d_2,q,a)$, where $Comp(d_1,p,a)$ and $Comp(d_2,q,a)$, are (graded) completions with respect to the parameter subgroup $< d_0 >$, that terminate in the subgroup $< d_0 >$, and $< e_1 >$ and $< e_2 >$ are abelian with pegs in $< d_0 >$ (i.e., abelian groups that commute with non-trivial elements in the terminal limit group $< d_0 >$).*

(4) *there exists a specialization $(x_1,\ldots,x_s,p,q,a)$ of the configuration limit group $Conf$, for which the corresponding elements (x_i,p,q,a) are distinct and rigid specializations of the rigid limit group, $Rgd(x,p,q,a)$ (strictly solid and belong to distinct strictly solid families), that can be extended to a specialization that factors through the duo limit group Duo (i.e., there exists a configuration homomorphism that can be extended to a specialization of Duo).*

Given a duo limit group, $Duo(d_1,p,d_2,q,d_0,a)$, and a specialization of the variables d_0, we call the set of specializations that factor through Duo for which the specialization of the variables d_0 is identical to the given one, a duo-family. We say that a duo family associated with a duo limit group Duo is covered by the duo limit groups $Duo_1,\ldots,Duo_t$, if there exists a finite collection of duo families associated with the duo limit groups, $Duo_1,\ldots,Duo_t$, and a covering closure of the duo family, so that each configuration homomorphism that can be extended to a specialization of a closure in the covering closure of the given duo family, can also be extended to a specialization that factors through one of the members of the finite collection of duo families of the duo limit groups $Duo_1,\ldots,Duo_t$ (see definition 1.16 in [Se2] for a covering closure).

In [Se9] we used the sieve procedure [Se6] to prove the existence of a finite collection of duo limit groups, that cover all the duo families associated with a duo limit group that is associated with a given rigid limit group.

Theorem 1.2 ([Se9],3.2). *Let F_k be a non-abelian free group, let s be a positive integer, and let $Rgd(x,p,q,a)$ be a rigid limit group defined over F_k. There exists a finite collection of duo limit groups associated with configuration homomorphisms of s distinct rigid homomorphisms of Rgd, $Duo_1,\ldots,Duo_t$, and some global bound b, so that every duo family that is associated with a duo limit group Duo, that is associated with configuration homomorphisms of s distinct rigid homomorphisms of Rgd, is covered by the given finite collection $Duo_1,\ldots,Duo_t$. Furthermore, every duo family that is associated with an arbitrary duo limit group Duo, is covered by at most b duo families that are associated with the given finite collection, $Duo_1,\ldots,Duo_t$.*

In this section we look for a partial generalization of theorem 1.2 to a general definable set. Given a definable set $L(p,q)$ we associate with it (canonically) a finite collection of graded limit groups (with respect to the parameter subgroup $< q >$). A "generic" point in each of these graded limit groups is contained in the definable set $L(p,q)$, and given a value q_0 of the variables q the (boundedly many) fibers that are associated with q_0 and the finite collection of graded limit groups, contain the projection $L(p,q_0)$ of the definable set $L(p,q)$. Later on we associate with this canonical collection of graded limit groups, a canonical collection of duo limit groups, and the obtained duo limit groups is the key tool in our classification

of imaginaries.

Theorem 1.3. *Let F_k be a non-abelian free group, and let $L(p,q)$ be a definable set over F_k. There exists a finite collection of graded limit groups, $G_1(z,p,q,a),\dots,G_t(z,p,q,a)$, (canonically) associated with $L(p,q)$, that we call the Diophantine Envelope of $L(p,q)$, for which:*

1. *For each j, $1 \leq j \leq t$, $G_j(z,p,q,a)$ is a graded completion (with respect to the parameter subgroup $< q,a >$. See definition 1.12 in [Se2] for a (graded) completion).*
2. *For each j, $1 \leq j \leq t$, there exists a test sequence $\{(z_n,p_n,q_0,a)\}$ of the completion $G_j(z,p,q,a)$, for which all the specializations $(p_n,q_0) \in L(p,q)$.*
3. *Given a specialization $(p_0,q_0) \in L(p,q)$, there exists an index j, $1 \leq j \leq t$, and a test sequence $\{(z_n,p_n,q_0,a)\}$ of the completion $G_j(z,p,q,a)$, for which all the specializations $(p_n,q_0) \in L(p,q)$, so that (p_0,q_0) can be extended to a specialization that factors through the same (graded) modular block of the completion $G_j(z,p,q,a)$ that contains the test sequence, $\{(z_n,p_n,q_0,a)\}$.*

Proof: Let $L(p,q)$ be a definable set. Recall that with a definable set $L(p,q)$ the sieve procedure associates a finite collection of graded (PS) resolutions that terminate in rigid and solid limit groups (with respect to the parameter subgroup $< p,q >$), and with each such graded resolution it associates a finite collection of graded closures that are composed from Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic collapse extra PS resolutions (see definitions 1.25-1.30 of [Se5] for the exact definitions).

By the sieve procedure [Se6], that eventually leads to quantifier elimination over a free group, the definable set $L(p,q)$ is equivalent to those rigid and strictly solid specializations of the terminal rigid and solid limit groups of the PS resolutions constructed along the sieve procedure, for which the PS resolutions associated with these rigid and solid specializations are not covered by the collection of Non-Rigid, Non-Solid, Left, Root and extra PS resolutions (minus the specializations that factor through the associated Generic collapse extra PS resolutions).

Therefore, using the output of the sieve procedure and the resolutions it constructs, with each terminating rigid or solid limit group $Term$ of a PS resolution along it we associate finitely many sets:

1. $B_1(Term)$ - the set of specializations of $< p >$ for which the terminal rigid or solid limit group $Term$ admits rigid or strictly solid specializations.
2. $B_2(Term)$ - the set of specializations of $< p >$ for which the associated Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through the associated Generic collapse extra PS resolutions), associated with the PS resolution that terminates in $Term$, form a covering closure of all the (ungraded) PS resolutions associated with the rigid or strictly solid specializations that are associated with the given specialization of $< p >$.
3. $B_3(Term)$ - the set of specializations of $< p >$ for which the Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through associated Generic collapse extra PS resolutions) form a covering closure of all the (ungraded) PS resolutions associated with a given specialization of $< p >$ and with PS resolutions that extend the PS

resolution that terminates in $Term$, and for which there exist strictly solid specializations of $Term$ with respect to that covering closure.

(4) $B_4(Term)$ - the set of specializations of $< p >$ in $B_3(Term)$, for which the associated Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through the associated Generic collapse extra PS resolutions), associated with the PS resolution that terminates in $Term$, form a covering closure of all the (ungraded) PS resolutions associated with the strictly solid specializations of $Term$, where these rigid and strictly solid specializations are associated with a given specialization of $< p >$, and are taken with respect to the covering closure of all the PS resolutions that extend the PS resolution that terminates in $Term$, that is associated with their collections of Non-Rigid, Non-Solid, Left, Root and extra PS resolutions.

Finally, using the sieve procedure, with a definable set $L(p, q)$ there are finitely many associated rigid and solid limit groups $Term_1, \ldots, Term_s$, so that $L(p, q)$ is the finite union:

$$L(p,q) \; = \; \cup_{i=1}^{s} \left(B_1(Term_i) \setminus B_2(Term_i)\right) \, \cup \, \left(B_3(Term_i) \setminus B_4(Term_i)\right).$$

We start the construction of the finite collection of graded limit groups that are associated with $L(p, q)$, by associating a finite collection of graded limit groups (that are graded with respect to $< q, a >$) with the sets $B_1(Term_i) \setminus B_2(Term_i)$, $i = 1, \ldots, s$.
Let $Term$ be one of the rigid or solid limit groups $Term_1, \ldots, Term_s$. The sieve procedure [Se6] associates with the terminal limit group $Term$, and the PS resolution that is associated with it, a finite collection of Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions (see sections 1 and 3 in [Se5] for the definition of these resolutions). Each of these associated resolutions is by construction a graded closure of the PS resolution that terminates in $Term$, and such a resolution terminates in a rigid or solid limit group (with respect to the parameter subgroup $< p, q >$).

Recall that by theorems 2.5 and 2.9 in [Se3], there exists a global bound on the number of rigid specializations of a rigid limit group, and a global bound on the number of strictly solid families of specializations of a solid limit group, for all possible specializations of the parameters subgroup. Hence, with each specialization of the parameter subgroup $< p, q >$, there are boundedly many rigid (strictly solid families of) specializations of the terminal limit group $Term$, and the the terminal limit groups of the resolutions that are associated with $Term$.

Given $Term$, we look at the collection of specializations of the form:

$$(x, y_1, \ldots, y_t, u_1, \ldots, u_m, v_1, \ldots, v_n, r, r_1, \ldots, r_n, p, q, a)$$

where:

(1) the integers t,m,n are bounded by the sum of the global bounds on the number of rigid and strictly solid families of specializations of the terminal rigid and solid limit groups of the resolutions that are associated with the terminal limit group $Term$.

(2) the specialization (x, p, q, a) is a rigid or a strictly solid specialization of the terminal rigid or solid limit group $Term$. The specializations (y_i, p, q, a),

$i = 1, \dots, t$, are rigid and strictly solid specializations of the terminal rigid and solid limit groups of the Non-Rigid, Non-Solid, Left, and Root PS resolutions that are associated with the PS resolution that terminates in $Term$. The rigid specializations are distinct and the strictly solid specializations belong to distinct strictly solid families, and the finite collection of specializations (y_i, p, q, a), $i = 1, \dots, t$, represent all the rigid and strictly solid families of specializations that are associated with (i.e., that extend) the rigid or strictly solid specialization (x, p, q, a).

(3) the specializations (v_j, p, q, a), $j = 1, \dots, n$, are distinct rigid and strictly solid specializations of the terminal (rigid and solid) limit groups of the Extra PS resolutions that are associated with $Term$ that extend the specialization (x, p, q, a). Furthermore, given the specialization (x, p, q, a), there are precisely n rigid or strictly solid families of specializations of the terminal rigid and solid limit groups of the Extra PS resolutions that extend the specialization (x, p, q, a).

(4) the specializations (u_j, p, q, a), $j = 1, \dots, m$, are distinct rigid and strictly solid specializations of the terminal (rigid and solid) limit groups of the Generic Collapse Extra PS resolutions that are associated with $Term$ that extend the specialization (x, p, q, a). Furthermore, given the specialization (x, p, q, a), there are precisely n rigid or strictly solid families of specializations of the terminal limit groups of Generic Collapse Extra PS resolutions that extend the specialization (x, p, q, a).

(5) the specializations r, include primitive roots of the edge groups in the graded abelian decomposition of the rigid or solid limit group $Term$ that are associated with the specialization (x, p, q, a), and they indicate what powers of the primitive roots are covered by the associated Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions (i.e., by the resolutions that are associated with the specializations y, u, and v).

(6) the specializations r_j, $j = 1, \dots, n$, include primitive roots of the edge groups in the graded abelian decompositions of the rigid or solid limit group terminal limit groups of the Extra PS resolutions that are associated with the specializations (v_j, p, q, a), $j = 1, \dots, n$, and they indicate what powers of the primitive roots are covered by the associated Generic Collapse Extra PS resolutions, i.e., by the resolutions that are associated with the specializations u.

The specializations (x, y, u, v, p, q, a) that satisfy properties (1)-(6) form "proof statements" for validation that $(p, q) \in L(p, q)$. By our standard methods (presented in section 5 of [Se1]), with this collection of specializations we can canonically associate a finite collection of graded limit groups (which is the Zariski closure of the collection). We view each of these (finitely many) limit groups, as graded with respect to the parameter subgroup $< q, a >$. We associate with each such limit group its graded taut Makanin-Razborov diagram (see proposition 2.5 in [Se4] for the construction of the taut Makanin-Razborov diagram), and with each resolution in the diagram we associate its (graded) completion.

We continue with each of the obtained graded completions in parallel. Given a graded completion, we associate with it the collection of sequences:

$$\{(b_\ell, z_\ell, x_\ell, y_\ell, v_\ell, u_\ell, r_\ell, p_\ell, q_\ell)\}$$

for which:

(1) $\{(z_\ell, x_\ell, y_\ell, v_\ell, u_\ell, r_\ell, p_\ell, q_\ell)\}$ is a test sequence of one of the obtained graded completions.

(2) for each index ℓ, $(b_\ell, p_\ell, q_\ell, a)$ is a rigid or a strictly solid specialization of one of the (rigid or solid) terminal limit groups of one of the Non-Rigid, Non-Solid, Left, Root, Extra PS, or Generic Collapse Extra PS resolutions that are associated with the specialization $(x_\ell, p_\ell, q_\ell, a)$, which is distinct from the rigid specializations and from the strictly solid families of specializations that are specified by the specialization:

$$(x_\ell, y_\ell, v_\ell, u_\ell, r_\ell, p_\ell, q_\ell)$$

Using the construction of graded formal limit groups that is presented in sections 2-3 in [Se2], we associate with the collection of sequences:

$$\{(b_\ell, z_\ell, x_\ell, y_\ell, v_\ell, u_\ell, r_\ell, p_\ell, q_\ell)\}$$

a (canonical) finite collection of (maximal formal) limit groups. By choosing each of the specializations $\{b_\ell\}$ to be a specialization for which the associated specialization, $(b_\ell, p_\ell, q_\ell, a)$ is the shortest in its strictly solid family, each of the maximal formal limit groups that is associated with the collection of sequences is in fact a graded closure of one of the graded completions we have started this step with.

We continue with each of the obtained graded closures in parallel. Given a graded closure, we associate with it the collection of specializations, $(c, t, b, z, x, y, u, v, r, p, q)$, for which:

(1) $(t, b, z, x, y, u, v, r, p, q, a)$ factors through the given graded closure.

(2) (c, b, p, q, a) demonstrates that the specialization (b, p, q, a), which is a specialization of the (rigid or solid) terminal limit group of one of the Non-Rigid, Non-Solid, Left, Root, Extra PS or Generic Collapse Extra PS resolutions that are associated with $Term$, is not rigid nor strictly solid or that it either coincides with one of the rigid specializations, or that it belongs to the strictly solid family of one the strictly solid specializations that is associated with the specialization: (z, x, y, u, v, p, q, a).

With the collection of specializations, $(c, t, b, z, x, y, u, v, r, p, q)$, we naturally associate (canonically) a collection of (maximal) limit groups (which is the Zariski closure of the collection). With these limit groups we associate a collection of graded taut resolutions (with respect to the subgroup $< q, a >$), that are constructed according to the first step of the sieve procedure (that is presented in [Se6]). Some of these resolutions are graded closures of the graded completion we have started with, and others have smaller complexity (in the sense of the complexities that are used along the sieve procedure). We continue to the next step with the completions of each of these resolutions that are not graded closures of the graded completion we have started the first step with.

We continue iteratively to the next steps, by first collecting test sequences with an extra rigid or strictly solid specialization of the terminal rigid and solid limit groups of the resolutions that are associated with $Term$, and then by collecting those specializations for which the extra specialization collapses, i.e., for which the extra specialization is demonstrated to be non-rigid or non-strictly solid or that it

belongs to the same family of specializations that is specified by the specializations (x,y,u,v,p,q,a) we have started with. At each step we analyze the collection of specializations using the construction that is used in the general step of the sieve procedure [Se6]. Since all the resolutions that are constructed along the obtained iterative procedure are the ones used in the sieve procedure, the iterative procedure terminates after finitely many steps, according to the proof of the termination of the sieve procedure that is given section 4 in [Se6] (theorem 22 in [Se6]).

So far we have associated graded limit groups with the subsets $B_1(Term_i) \setminus B_2(Term_i)$, $i=1,\dots,s$. Let $Term$ be one of the terminal solid limit groups $Term_i$, $1 \leq i \leq s$. We associate finitely many graded limit groups (with respect to the parameter subgroup $< q,a >$) with the subset $B_3(Term) \setminus B_4(Term)$ in a similar way to what we did with the set $B_1(Term) \setminus B_2(Term)$.

Let $Term_1,\dots,Term_b$ be the rigid and solid limit groups that appear in the (taut) graded Makanin-Razborov diagram of $Term$ (with respect to the parameter subgroup $< p,q,a >$). By theorems 2.5, 2.9, and 2.13 in [Se3] there exists a global bound on the number of rigid specializations of a rigid limit group, and a global bound on the number of strictly solid families of specializations of a solid limit group, even with respect to a given covering closure (see theorem 2.13 in [Se3]), for all possible specializations of the parameters subgroup.

Hence, given the rigid and solid limit groups $Term_1,\dots,Term_b$, and the terminal rigid and solid limit groups of the Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions associated with them, there is a global bound on the number of distinct rigid and strictly solid families of specializations that are associated with a given specialization of the parameter subgroup $< p,q,a >$, and these terminal rigid and solid limit groups.

We go over all the specializations of the form:

$$(x,x_1,\dots,x_h,y_1,\dots,y_t,u_1,\dots,u_m,v_1,\dots,v_n,r,r_1,\dots,r_n,p,q,a)$$

where:

(1) the integers h,t,m,n are bounded by the sum of the global bounds on the number of rigid and strictly solid families of specializations (with respect to the possible covering closures) of the terminal rigid and solid limit groups $Term_1,\dots,Term_b$, and of the terminal rigid and solid limit groups of the resolutions that are associated with the terminal limit group $Term$ and $Term_1,\dots,Term_b$.

(2) the specialization (x,p,q,a) is a rigid or a strictly solid specialization of the terminal rigid or solid limit group $Term$.

(2) the specializations (x_i,p,q,a), $i=1,\dots,h$, are either rigid and strictly solid specializations of the rigid or solid limit groups $Term_1,\dots,Term_b$, or they are strictly solid with respect to the an associated covering closure (that is specified by the specialization itself, i.e., by the specializations y,u,v). Furthermore, the finite collection of specializations (x_i,p,q,a), $i=1,\dots,h$, represent all the rigid and strictly solid families of specializations (and the strictly solid ones with respect to covering closures) that are associated with the given specialization of the subgroup $< p,q >$, and the terminal limit groups $Term_1,\dots,Term_b$.

(3) the specializations (y_i,p,q,a), $i=1,\dots,t$, are rigid and strictly solid specializations of the terminal rigid and solid limit groups of the Non-Rigid,

Non-Solid, Left, and Root PS resolutions that are associated with the PS resolution that terminates in $Term$ and $Term_1, \dots, Term_b$. The rigid specializations are distinct and the strictly solid specializations belong to distinct strictly solid families, and the finite collection of specializations (y_i, p, q, a), $i = 1, \dots, t$, represent all the rigid and strictly solid families of specializations that are associated with (i.e., that extend) the strictly solid specialization (with respect to the associated covering closure) (x, p, q, a), and the rigid and strictly solid specializations (x_i, p, q, a), $i = 1, \dots, h$.

(4) the specializations (v_j, p, q, a), $j = 1, \dots, n$, are distinct rigid and strictly solid specializations of the terminal (rigid and solid) limit groups of the Extra PS resolutions that are associated with $Term$ that extend the specialization (x, p, q, a) and (x_i, p, q, a), $i = 1, \dots, h$. Furthermore, given the specializations (x, p, q, a) and (x_i, p, q, a), $i = 1, \dots, h$, there are precisely n rigid or strictly solid families of specializations of the terminal rigid and solid limit groups of the Extra PS resolutions that extend the specialization (x, p, q, a).

(5) the specializations (u_j, p, q, a), $j = 1, \dots, m$, are distinct rigid and strictly solid specializations of the terminal (rigid and solid) limit groups of the Generic Collapse Extra PS resolutions that are associated with $Term$ and $Term_1, \dots, Term_b$, that extend the specializations (x, p, q, a) and (x_i, p, q, a), $i = 1, \dots, h$. Furthermore, given the specialization (x, p, q, a) and (x_i, p, q, a), $i = 1, \dots, h$, there are precisely n rigid or strictly solid families of specializations of the terminal limit groups of Generic Collapse Extra PS resolutions that extend the specialization (x, p, q, a).

(6) the specializations r, include primitive roots of the edge groups in the graded abelian decompositions of the rigid or solid limit groups $Term$ and $Term_1, \dots, Term_b$, that are associated with the specializations (x, p, q, a) and (x_i, p, q, a), $i = 1, \dots, h$, and they indicate what powers of the primitive roots are covered by the associated Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions (i.e., by the resolutions that are associated with the specializations y, u, and v).

(6) the specializations r_j, $j = 1, \dots, n$, include primitive roots of the edge groups in the graded abelian decompositions of the rigid or solid limit group terminal limit groups of the Extra PS resolutions that are associated with the specializations (v_j, p, q, a), $j = 1, \dots, n$, and they indicate what powers of the primitive roots are covered by the associated Generic Collapse Extra PS resolutions, i.e., by the resolutions that are associated with the specializations u.

(7) the specialization (x, p, q, a) is not a strictly solid specialization of $Term$, but it is strictly solid with respect to the covering closure associated with the specializations y, u, and v (see definition 2.12 in [Se3] for a strictly solid specialization with respect to a covering closure).

With this collection of specializations we can canonically associate a finite collection of graded limit groups (which is the Zariski closure of the collection). We view each of these (finitely many) limit groups, as graded with respect to the parameter subgroup $< q, a >$. We associate with each such limit group its graded taut Makanin-Razborov diagram, and with each resolution in the diagram we associate its (graded) completion.

We continue as we did in assigning a finite collection of limit groups with the sets $B_1(Term) \setminus B_2(Term)$. At each step we first collect all the test sequences for which there exists an extra rigid or strictly solid specialization of one of the terminal limit groups $Term_1, \ldots, Term_b$ or of one of the terminal limit groups of the resolutions that are associated with $Term$ and $Term_1, \ldots, Term_b$, i.e., a rigid or strictly solid specialization which is distinct from the rigid and strictly solid families that are specified by the specializations of the test sequence. We analyze the obtained collection of sequences of specializations using the analysis that is used in the construction of (graded) formal limit groups in section 3 in [Se2]. Then we collect all the specializations for which the extra rigid or strictly solid specialization collapses, i.e., is not rigid nor strictly solid or it belongs to the rigid and strictly solid families that are specified by the given specializations, analyze the obtained collection using the analysis that is used in the general step of the sieve procedure, and finally continue to the next step only with those (graded) resolutions that are not graded closures of the completions we have started the current step with. By the termination of the sieve procedure (theorem 22 in [Se6]), the obtained procedure terminates after finitely many steps.

The procedures that we used so far, that are associated with the sets $B_1(Term_i) \setminus B_2(Term_i)$, and the sets $B_3(Term_i) \setminus B_4(Term_i)$, $i = 1, \ldots, s$, construct finitely many graded completions that are associated with the definable set we have started with, $L(p,q)$. These are the graded completions with which we have started each procedure, and the completions and closures of the completions of the developing resolutions that are constructed in each step of the terminating iterative procedures (see the general step of the sieve procedure in [Se6], for the construction of the developing resolution).

Let $G_1(z,p,q,a), \ldots, G_{t'}(z,p,q,a)$ be the graded completions that are associated with $L(p,q)$ according to the procedures presented above. We look at the subset (up to a change of order of the original set of graded completions) $G_1(z,p,q,a), \ldots, G_t(z,p,q,a)$, for some $1 \leq t \leq t'$, for which for each j, $1 \leq j \leq t$, there exists a test sequence $\{z_n, p_n, q, a\}$ of the completion $G_j(z,p,q,a)$, for which all the specializations $(p_n, q) \in L(p,q)$.

The collection of graded completions, $G_1(z,p,q,a), \ldots, G_t(z,p,q,a)$, clearly satisfies part (1) and (2) of theorem 1.3. The iterative construction that leads to their construction guarantees that they satisfy part (3) as well.

□

Given a definable set, $L(p,q)$, theorem 1.3 associates with it a Diophantine envelope. Starting with the Diophantine envelope of a definable set, we further associate with it a *Duo Envelope*, that is the main tool that we use in classifying imaginaries over free and hyperbolic groups.

Theorem 1.4. *Let F_k be a non-abelian free group, let $L(p,q)$ be a definable set over F_k, and let $G_1(z,p,q,a), \ldots, G_t(z,p,q,a)$ be the Diophantine envelope of $L(p,q)$ (see theorem 1.3). There exists a finite collection of duo limit groups:*

$$Duo_1(d_1,p,d_2,q,d_0,a), \ldots, Duo_r(d_1,p,d_2,q,d_0,a)$$

which is (canonically) associated with $L(p,q)$, that we call the Duo Envelope of $L(p,q)$, for which:

(1) *For each index i, $1 \leq i \leq r$, there exists a test sequence of the duo limit*

group Duo_i, that restricts to a sequence of couples $\{(p_n, q_n)\}$, so that for every index n, $(p_n, q_n) \in L(p, q)$.

(2) *A (duo) test sequence $\{(d_1(n), p_n, d_2(n), q_n, d_0, a)\}$ of each of the duo limit groups $Duo_1, \dots, Duo_r$ restricts to a sequence $\{p_n\}$ that can be extended to a test sequence of at least one of the graded completions $G_1, \dots, G_t$.*

(3) *Given a test sequence $\{(z_n, p_n, q_0, a)\}$ of one of the graded completions, $G_1, \dots, G_t$, for which $(p_n, q_0) \in L(p, q)$ for every index n, there exists a subsequence of the sequence $\{p_n\}$, so that the sequence of couples (still denoted) $\{(p_n, q_0)\}$ can be extended to a test sequence of specializations of at least one of the duo limit groups $Duo_1, \dots, Duo_r$.*

(4) *Given a specialization $(p_0, q_0) \in L(p, q)$, there exists an index i, $1 \leq i \leq r$, and a duo family of the duo limit group Duo_i, so that (p_0, q_0) extends to a specialization that factors through the duo family, and there exists a test sequence of the duo family that restricts to specializations $\{(p_n, q_n)\}$, so that for every index n, $(p_n, q_n) \in L(p, q)$.*

(5) *Let Duo be some duo limit group of $L(p, q)$. Every duo family of Duo that admits a test sequence that restricts to a sequence of specializations $\{(p_n, q_n)\}$, for which $(p_n, q_n) \in L(p, q)$ for every index n, and for which the sequence (p_n, q_n) can be extended to a test sequence of one the graded completions, $G_1, \dots, G_t$, is boundedly covered by the Duo envelope, $Duo_1, \dots, Duo_r$. i.e., there exists some constant $b > 0$ (that depends only on the duo limit group Duo), for which given a duo family of the duo limit group Duo, there exist at most b duo families of the Duo envelope, $Duo_1, \dots, Duo_r$, so that given an arbitrary test sequence that factors through the given duo family of the duo limit group Duo, and restricts to specializations $(p_n, q_n) \in L(p, q)$, so that the sequence $\{(p_n, q_n)\}$ can be extended to specializations that form a test sequence of one of the graded completions, $G_1, \dots, G_t$, there exists a subsequence of the sequence $\{(p_n, q_n)\}$ that can be extended to specializations of one of the (boundedly many) duo families that are associated with the Duo envelope $Duo_1, \dots, Duo_r$.*

Proof: When we dealt with a Diophantine set (section 2 in [Se9]), or with a rigid or solid limit group (section 3 in [Se9]), from the directed diagram that is associated with these groups and the graded completions that are associated with its vertices, it was immediate to obtain a canonical collection of duo limit groups that cover any duo family that is associated with the Diophantine set or with the rigid or solid limit group (see section 3 in [Se9]). When we deal with a general definable set $L(p, q)$ the construction of such a universal family of duo limit groups that is associated with its Diophantine envelope is more involved.

We will start with the same construction that was applied in constructing the duo limit groups that are associated with a rigid or a solid limit group in section 3 of [Se9], and then iteratively refine this construction using the sieve method [Se6], in a similar way to the construction of the Diophantine envelope (theorem 1.3).

Let $G_1, \dots, G_t$ be the Diophantine envelope of the definable set $L(p, q)$ (see theorem 1.3). We start with the graded completions $G_1, \dots, G_t$ in parallel. With each graded completion G_j, $1 \leq j \leq t$, we first associate a finite collection of duo limit groups (that are not yet necessarily part of the duo envelope).

To construct these duo limit groups, we look at the entire collection of graded test sequences that factor through the given graded completion, G_j, for which the (re-

stricted) sequence of specializations $\{p_n\}$ can be extended to specializations of one of the graded completions (the Diophantine envelope) $G_1(z,p,q,a)\dots,G_t(z,p,q,a)$. With this entire collection of graded test sequences, and their extensions to specializations of $G_1,\dots,G_t$, we associate finitely many graded Makanin-Razborov diagrams, precisely as we did in constructing the formal graded Makanin-Razborov diagram in section 3 of [Se2]. As in the formal Makanin-Razborov diagram, each resolution in the diagrams we construct terminates with a (graded) closure of the given graded completion, G_j, we have started with, amalgamated with another group along its base (which is the terminal rigid or solid limit group of the graded completion G_j), and the abelian vertex groups that commute with non-trivial elements in the base.

By construction, a completion of a resolution in one of the constructed graded diagrams is a duo limit group. We take the completions of the resolutions that appear in the finitely many diagrams that are associated with the graded completion G_j, to be the preliminary (finite) collection of duo limit groups that are associated with G_j.

We proceed by constructing an iterative procedure that is similar to the one used in the proof of theorem 1.3 to construct the Diophantine envelope, and is based on the sieve procedure [Se6]. At each step we first collect all the test sequences of the current duo limit group (that is associated with G_j), for which for every specialization from the sequence there exists an extra rigid or strictly solid specialization, and analyze the obtained collection of sequences of specializations using the analysis of (graded) formal limit groups that appears in sections 2 and 3 in [Se2]. Then we collect all the test sequences of that graded completion G_j that can be extended to specializations of the obtained cover of the current duo limit group, for which the extra rigid or strictly solid specialization collapses. We analyze the obtained collection using the analysis that is used in the general step of the sieve procedure, and finally continue to the next step only with those (graded, formal) resolutions that are not graded closures of the duo limit group we have started the current step with. By the termination of the sieve procedure (theorem 22 in [Se6]), the obtained procedure terminates after finitely many steps.

Finally, the duo limit groups that we proceed with, from which we can choose a subcollection that satisfies the properties that are listed in Theorem 1.4, are those duo limit groups that are obtained along the sieve procedures that were constructed for all the (finitely many) duo limit groups that were associated with the graded completion (the Diophantine envelope) $G_1,\dots,G_t$.

We set the Duo envelope of the definable set $L(p,q)$, $Duo_1,\dots,Duo_r$, to be those duo limit groups that are associated with the Diophantine envelope, $G_1,\dots,G_t$, for which there exists a duo family having a test sequence, so that all the specializations in the test sequence restrict to elements (p,q) in $L(p,q)$ (i.e., in particular, a "generic point" in Duo_i is in $L(p,q)$).

The iterative procedure that constructs the collection of duo limit groups, $Duo_1,\dots,Duo_r$, guarantees that they satisfy properties (1)-(3) of theorem 1.4. Part (4) follows from part (3) together with part (3) of theorem 1.3. To prove part (5) we assume that Duo is a duo limit group, and that there exists a test sequence of Duo that restricts to specializations (p_n,q_n), so that:

(i) the sequence (p_n,q_n) can be extended to a test sequence of one of the graded completions, $G_1,\dots,G_t$.

(ii) for every index n, $(p_n, q_n) \in L(p,q)$.

We look at the collection of all the test sequences of the given duo limit group Duo that satisfy properties (i) and (ii). By properties (i) and (ii) and the construction of the diagram that was used to construct the duo envelope, $Duo_1, \dots, Duo_r$, given such a test sequence, its sequence of restrictions $\{(p_n, q_n)\}$ can be extended to specializations that factor through one of the duo limit groups that are constructed in the initial step of the diagram (that was used to construct the duo envelope). By property (i) and the construction of the diagram, given such a test sequence it has a subsequence so that the sequence of restrictions (still denoted) $\{(p_n, q_n)\}$ of the subsequence satisfies one of the following:

(1) $\{(p_n, q_n)\}$ can be extended to specializations that factor through one of the duo limit groups, $Duo_1, \dots, Duo_r$, that are associated with the initial step of the diagram that was used to construct the duo envelope.

(2) $\{(p_n, q_n)\}$ can be extended to specializations that factor through one of the developing resolutions that are associated with the second step of the diagram that was used to construct the duo envelope.

If we continue with this argument iteratively, we obtain:

Lemma 1.5. *Given a test sequence that satisfies properties (i) and (ii), it has a subsequence that its sequence of projections, $\{(p_n, q_n)\}$, can be extended to a sequence of specializations that factor through one of the duo limit groups, $Duo_1, \dots, Duo_r$.*

Having proved lemma 1.5, we continue with all the test sequences of Duo that satisfy properties (i) and (ii), and for which their projections $\{(p_n, q_n)\}$ can be extended to a sequence of specializations that factor through one of the duo limit groups, $Duo_1, \dots, Duo_r$.

With each test sequence of Duo from this collection, and its projection $\{(p_n, q_n)\}$, we consider all the shortest extensions of the sequence $\{(p_n, q_n\}$ to specializations of one of the duo limit groups, $Duo_1, \dots, Duo_r$. By the techniques presented in section 3 of [Se2] to analyze graded formal limit groups, with this collection of shortest extensions we can canonically associate finitely many maps from the collection of duo limit groups, $Duo_1, \dots, Duo_r$ to (finitely many) closures of the given duo limit group Duo.

The existence of these finitely many maps, and finitely many closures, guarantees that the sequence of projections $\{(p_n, q_n)\}$ of a test sequence of one of the specified closures of Duo can be extended to a sequence of specializations of one of the duo limit groups, $Duo_1, \dots, Duo_r$. Furthermore, a duo family of Duo is covered by boundedly many duo families of the specified closures of Duo, and each duo family of one of the specified closures is covered by boundedly many duo families of the duo envelope, $Duo_1, \dots, Duo_r$, and part (5) of theorem 1.4 follows.

□

§2. Few Basic Imaginaries

Our goal in this paper is to study definable equivalence relations over free and hyperbolic groups. Before we analyze the structure of a general definable equivalence relation over such groups, we introduce some basic well-known definable equivalence relations over a free (or hyperbolic) group, and prove, using Diophantine envelopes

that were presented in the first section, that these equivalence relations are imaginaries (non-reals). Later on we show that if one adds these basic imaginaries as new sorts to the model of a free or a hyperbolic group, then for any definable equivalence relation there exists a definable multi-function that separates classes and maps every equivalence class into a uniformly bounded set, i.e., that adding the basic imaginaries as sorts geometrically eliminates imaginaries over free and hyperbolic groups. We start by proving that conjugation is an imaginary.

Theorem 2.1. *Let $F_k =< a_1, \ldots, a_k >$ be a non-abelian free group, and let:*

$$Conj(x_1, x_2) \;=\; \{\, (x_1, x_2) \,|\, \exists u \;\, ux_1u^{-1} = x_2 \,\}$$

be the definable equivalence relation associated with conjugation over F_k. Then $Conj(x_1, x_2)$ is an imaginary.

Proof: To prove that conjugation is an imaginary, we need to show that there is no positive integer m and a (definable) function: $f : F_k \to F_k^m$ that maps each conjugacy class in F_k to a point, and distinct conjugacy classes to distinct points in F_k^m. To prove that there is no such function, we use the precise (geometric) description of a definable set that was obtained using the sieve procedure for quantifier elimination presented in [Se6] (the same description that we used in proving the stability of free and hyperbolic groups in [Se8]), and the Diophantine envelope of a definable set that was constructed in theorem 1.3.

Recall that by the output of the sieve procedure [Se6], with a definable set $L(p)$ there are associated rigid and solid limit groups (with respect to the parameter subgroup $< p >$), $Term_1, \ldots, Term_s$, and with each limit group, $Term_i$, there are associated sets $B_j(Term_i)$, $j = 1, \ldots, 4$, that are described in the first section, so that $L(p)$ is the finite union:

$$L(p) \;=\; \cup_{i=1}^{s} \left(B_1(Term_i) \setminus B_2(Term_i)\right) \cup \left(B_3(Term_i) \setminus B_4(Term_i)\right).$$

Suppose that conjugation is not an imaginary. Then there exists a definable function: $f : F_k \to F_k^m$, that maps each conjugacy class to a point, and different conjugacy classes to distinct points. Let $L(v, x_1, \ldots, x_m)$ be the graph of the function f, that by our assumption on the definability of f has to be a definable set, for which:

(i) for every possible value of the variable v there exists a unique value of the tuple, $(x_1, \ldots, x_m)$.

(ii) the value of the tuple $(x_1, \ldots, x_m)$ is the same for elements v in the same conjugacy class, and distinct for different conjugacy classes.

Using theorem 1.3, we associate with the definable set $L(v, x_1, \ldots, x_m)$, its Diophantine envelope with respect to the parameter v (i.e., in the statement of theorem 1.3, we set v to be q, and $(x_1, \ldots, x_m)$ to be p). Let $G_1, \ldots, G_t$ be the graded completions that form this Diophantine envelope. Since for every possible value of v there exists a unique value of $x_1, \ldots, x_m$ in $L(v, x_1, \ldots, x_m)$, each of the graded completions, $G_j(z, x_1, \ldots, x_m, v, a)$, is either a rigid limit group (with respect to the parameters v), or it is a solid limit group where the subgroup $< x_1, \ldots, x_m, v, a >$ is contained in the distinguished vertex group (that is stabilized by $< v, a >$).

With the definable set $L(v, x_1, \ldots, x_m)$ we associate three sequences of specializations: $\{u_n\}$, $\{(v_n, x_1^n, \ldots, x_m^n)\}_{n=1}^{\infty}$ and $\{(\hat{v}_n, \hat{x}_1^n, \ldots, \hat{x}_m^n)\}_{n=1}^{\infty}$, so that:

(1) for every index n, the tuples $(v_n, x_1^n, \ldots, x_m^n)$ and $(\hat{v}_n, \hat{x}_1^n, \ldots, \hat{x}_m^n)$, are in the definable set $L(v, x_1, \ldots, x_m)$.

(2) v_n and u_n are taken from a test sequence in the free group F_k (see theorem 1.1 and lemma 1.21 in [Se2] for a test sequence), and $\hat{v}_n = u_n v_n u_n^{-1}$.

(3) $2 \cdot |u_n| > |v_n| > \frac{1}{2} \cdot |u_n|$, where $|w|$ is the length of the word $w \in F_k$, with respect to a fixed generating set of F_k.

By theorem 1.3, each of the specializations $(v_n, x_1^n, \ldots, x_m^n)$ extends to a specialization that factors through one of the graded completions, $G_1, \ldots, G_t$, that form the Diophantine envelope of the definable set $L(v, x_1, \ldots, x_m)$. By passing to a subsequence, and changing the order of the graded completions, we can assume that they all factor through the graded completion G_1. Since G_1 is rigid or solid with respect to the parameter subgroup $< v, a >$, and since the sequence $\{v_n\}$ was chosen as a part of a test sequence, we can pass to a further subsequence, so that:

(1) there exists a retraction $\eta : G_1 \to H =< v, a >=< v > * F_k$.

(2) for each index n, there is a retraction, $\nu_n : H =< v, a > \to < a >= F_k$, given by: $\nu_n(v) = v_n$, and for every index i, $1 \leq i \leq m$, $x_i^n = \nu_n \circ \eta(x_i)$.

(3) by the construction of the Diophantine envelope, the graded completion G_1 contains elements that together are supposed to validate that the elements $(v, x_1, \ldots, x_m)$ are indeed elements of the definable set $L(v, x_1, \ldots, x_m)$ (see the construction of the Diophantine envelope in theorem 1.3). For each index n, the restriction of the composition $\nu_n \circ \eta : G_1 \to F_k$ to these elements validates that $(v_n, x_1^n, \ldots, x_m^n) \in L(v, x_1, \ldots, x_m)$.

By (1)-(3) and since $\hat{v}_n = u_n v_n u_n^{-1}$, and u_n and v_n are taken from a test sequence, if we set $\hat{\nu}_n : H \to F_k$ to be the retraction given by: $\hat{\nu}_n(v) = \hat{v}_n = u_n v_n u_n^{-1}$, then after possibly passing to a further subsequence, for every index n and every index i, $1 \leq i \leq m$: $\hat{x}_i^n = \hat{\nu}_n \circ \eta(x_i)$.
Since the elements $x_1, \ldots, x_m$ are distinct for distinct conjugacy classes of specializations of v, and the elements v_n are not conjugate, the tuples $(x_1^n, \ldots, x_m^n)$ are distinct for distinct indices n. Hence, there exists an index i, $1 \leq i \leq m$, for which $\eta(x_i) \notin F_k$. Therefore, for large enough n:

$$x_i^n = \nu_n \circ \eta(x_i) \neq \hat{\nu}_n \circ \eta(x_i) = \hat{x}_i^n.$$

But, v_n is conjugate to $\hat{v}_n$, and both tuples $(v_n, x_1^n, \ldots, x_m^n)$ and $(\hat{v}_n, \hat{x}_1^n, \ldots, \hat{x}_m^n)$ are contained in the definable set $L(v, x_1, \ldots, x_m)$. Hence, for every index n, and every i, $1 \leq i \leq m$, $x_i^n = \hat{x}_i^n$, and we get a contradiction.

□

Having proved that conjugation is an imaginary, we further show that left and right cosets of cyclic subgroups are imaginaries.

Theorem 2.2. *Let $F_k =< a_1, \ldots, a_k >$ be a non-abelian free group, let m be a positive integer and let:*

$$Left(x_1, y_1, x_2, y_2) = \{ (x_1, y_1, x_2, y_2) \mid y_1, y_2 \neq 1 \wedge [y_1, y_2] = 1 \wedge \exists y \, [y, y_1] = 1 \wedge x_1^{-1} x_2 = y^m \}$$

$$Right(x_1, y_1, x_2, y_2) = \{ (x_1, y_1, x_2, y_2) \mid y_1, y_2 \neq 1 \wedge [y_1, y_2] = 1 \wedge \exists y \, [y, y_1] = 1 \wedge x_1 x_2^{-1} = y^m \}$$

be the definable equivalence relation associated with left and right cosets of cyclic subgroups over F_k. Then Left and Right are imaginaries.

Proof: The proof we give is similar to the one we used in theorem 2.1. Suppose that Left is not an imaginary. Then there exists a definable function: $f : F_k^2 \to F_k^r$, that maps each left coset (of the corresponding cyclic subgroup) to a point, and different left cosets to distinct points. Let $L(t, s, x_1, \ldots, x_r)$ be the definable set associated with the definable function $f(t, s)$. Note that for every possible specialization (t_0, s_0) of (t, s), there exists a unique specialization $(t_0, s_0, x_1^0, \ldots, x_r^0)$ that belongs to the set $L(t, s, x_1, \ldots, x_r)$, and if $(t_0, s_0, x_1^0, \ldots, x_r^0) \in L(t, s, x_1, \ldots, x_r)$ then $(t_0 s_0^{\ell m}, s_0, x_1^0, \ldots, x_r^0) \in L(t, s, x_1, \ldots, x_r)$ for every integer ℓ.

We proceed as in the proof of theorem 2.1. Using theorem 1.3, we associate with the definable set $L(t, s, x_1, \ldots, x_m)$, its Diophantine envelope with respect to the parameter subgroup $< t, s >$ (i.e., in the statement of theorem 1.3, we set t, s to be q, and $x_1, \ldots, x_m$ to be p). Let $G_1, \ldots, G_t$ be the graded completions that form this Diophantine envelope. Since for every possible value of the couple (t, s) there exists a unique value of $x_1, \ldots, x_r$ in $L(t, s, x_1, \ldots, x_r)$, each of the graded completions, $G_j(z, x_1, \ldots, x_r, t, s, a)$, is either a rigid limit group (with respect to the parameter subgroup $< t, s, a >$), or it is a solid limit group where the subgroup $< x_1, \ldots, x_r, t, s, a >$ is contained in the distinguished vertex group (that is stabilized by $< t, s, a >$).

With the definable set $L(t, s, x_1, \ldots, x_r)$ we associate two sequences of specializations: $\{(t_n, s_n, x_1^n, \ldots, x_r^n)\}_{n=1}^{\infty}$ and $\{(\hat{t}_n, s_n, \hat{x}_1^n, \ldots, \hat{x}_r^n)\}_{n=1}^{\infty}$, so that:

(1) for every index n, the tuples $(t_n, s_n, x_1^n, \ldots, x_r^n)$ and $(\hat{t}_n, s_n, \hat{x}_1^n, \ldots, \hat{x}_r^n)$, are in the definable set $L(t, s, x_1, \ldots, x_r)$.
(2) t_n and s_n are taken from a test sequence in the free group F_k (see theorem 1.1 and lemma 1.21 in [Se2] for a test sequence), and $\hat{t}_n = t_n s_n^m$.
(3) $2 \cdot |s_n| > |t_n| > \frac{1}{2} \cdot |s_n|$, where $|w|$ is the length of the word $w \in F_k$, with respect to a fixed generating set of F_k.

By theorem 1.3, each of the specializations $(t_n, s_n, x_1^n, \ldots, x_r^n)$ extends to a specialization that factors through one of the graded completions, $G_1, \ldots, G_t$, that form the Diophantine envelope of the definable set $L(t, s, x_1, \ldots, x_m)$. By passing to a subsequence, and changing the order of the graded completions, we can assume that they all factor through the graded completion G_1. Since G_1 is rigid or solid with respect to the parameter subgroup $< t, s, a >$, and since the sequences, $\{t_n\}$ and $\{s_n\}$ were chosen as a part of a test sequence, we can pass to a further subsequence, so that:

(1) there exists a retraction $\eta : G_1 \to H =< t, s, a >=< t > * < s > * F_k$.
(2) for each index n, there is a retraction, $\nu_n : H =< t, s, a > \to < a >= F_k$, given by: $\nu_n(t) = t_n$, $\nu_n(s) = s_n$, and for every index i, $1 \leq i \leq r$, $x_i^n = \nu_n \circ \eta(x_i)$.
(3) by the construction of the Diophantine envelope, the graded completion G_1 contains elements that together are supposed to validate that the elements $(t, s, x_1, \ldots, x_r)$ are indeed elements of the definable set $L(t, s, x_1, \ldots, x_r)$ (see the construction of the Diophantine envelope in theorem 1.3). For each index n, the restriction of the composition $\nu_n \circ \eta : G_1 \to F_k$ to these elements validates that $(t_n, s_n, .x_1^n, \ldots, x_r^n) \in L(t, s, x_1, \ldots, x_r)$.

By (1)-(3) and since $\hat{t}_n = t_n s_n^m$, and t_n and s_n are taken from a test sequence,

if we set $\hat{\nu}_n : H \to F_k$ to be the retraction given by: $\hat{\nu}_n(t) = t_n s_n^m$ and $\hat{\nu}_n(s) = s_n$, then after possibly passing to a further subsequence, for every index n and every index i, $1 \leq i \leq r$: $\hat{x}_i^n = \hat{\nu}_n \circ \eta(x_i)$.
Since the elements $x_1, \ldots, x_r$ are distinct for distinct left cosets, and the elements t_n belong to distinct left cosets of the cyclic subgroups $s_{n'}^m$, there exists an index i, $1 \leq i \leq r$, for which $\eta(x_i) \notin < s, a >$. Therefore, for large enough n:

$$x_i^n = \nu_n \circ \eta(x_i) \neq \hat{\nu}_n \circ \eta(x_i) = \hat{x}_i^n.$$

But, t_n is in the same left coset as $\hat{t}_n = t_n s_n^m$, and both tuples $(t_n, s_n, x_1^n, \ldots, x_r^n)$ and $(\hat{t}_n, s_n, \hat{x}_1^n, \ldots, \hat{x}_r^n)$ are contained in the definable set $L(t, s, x_1, \ldots, x_r)$. Hence, for every index n, and every i, $1 \leq i \leq r$, $x_i^n = \hat{x}_i^n$, and we get a contradiction.

□

Having Proved that cosets of cyclic groups are imaginaries, we further show that double cosets of cyclic groups are imaginaries as well.

Theorem 2.3. *Let $F_k =< a_1, \ldots, a_k >$ be a non-abelian free group, let m_1, m_2 be positive integers and let:*

$$Dcoset(y_1, x_1, z_1, y_2, x_2, z_2) \ = \ \{ \, (y_1, x_1, z_1, y_2, x_2, z_2) \mid y_1, z_1, y_2, z_2 \neq 1 \wedge [y_1, y_2] = [z_1, z_2] = 1 \wedge$$

$$\wedge \exists y, z \, [y, y_1] = [z, z_1] = 1 \wedge y^{m_1} x_1 z^{m_2} = x_2 \, \}$$

be the definable equivalence relation associated with double cosets of cyclic subgroups over F_k. Then Dcoset is an imaginary.

Proof: A straightforward modification of the proof for left and right cosets (theorem 2.2), proves that double cosets are imaginaries as well.

□

So far we proved that 3 basic definable equivalence relations over a free group are imaginaries. The arguments that were used to prove them generalize to an arbitrary non-elementary torsion-free hyperbolic group using the results of [Se8].

Theorem 2.4. *Let Γ be a non-elementary, torsion-free (Gromov) hyperbolic group. Then conjugation, left and right cosets and double cosets of cyclic subgroups (see theorems 2.1-2.3) are imaginaries over Γ.*

§3. Separation of Variables

In section 3 of [Se9] we have introduced Duo limit groups, and associated a finite collection of Duo limit groups with a given rigid or solid limit group. In the first section of this paper we have constructed the Diophantine envelope of a definable set (theorem 1.3), and then used it to construct the Duo envelope of a definable set (theorem 1.4).

Recall that by its definition (see definition 1.1), a Duo limit group *Duo* admits an amalgamated product: $Duo =< d_1, p > *_{<d_0, e_1>} < d_0, e_1, e_2 > *_{<d_0, e_2>} < d_2, q >$ where $< e_1 >$ and $< e_2 >$ are free abelian groups with pegs in $< d_0 >$, i.e., free abelian groups that commute with non-trivial elements in $< d_0 >$. A specialization of the parameters $< d_0 >$ of a Duo limit group gives us a Duo family of it.

To analyze definable equivalence relations over a free (or a hyperbolic) group, we will need to further study the parameters ($< d_0 >$) that are associated with the Duo families that are associated with the Duo limit groups that form the Duo envelope of a definable equivalence relation. To do that we will need to get a better "control" or understanding of the parameters $< d_0 >$ that are associated with a duo family and then with an equivalence class of a given equivalence relation.

In this section we modify and further analyze the construction of the Duo envelopes that were presented in theorem 1.4, in the special case of a definable equivalence relation, to carefully study the set of values of the parameters that are associated with the duo families that are associated with each equivalence class. This careful study, that uses what we call $uniformization$ limit groups that we associate with the Duo envelope, enables one to associate a "bounded" set of (values of) certain subgroups of the parameters that are associated with the Duo families of the Duo envelope, for each equivalence class of a definable equivalence relation (the bounded set of values of the subgroups of parameters is modulo the basic imaginaries that were presented in the previous section).

The bound that we achieve on the number of specializations of the subgroups that we look at, allows us to obtain what we view as "separation of variables". This means that with the original subgroups of parameters, $< p >$ and $< q >$, we associate a bigger subgroup, for which there exists a graph of groups decomposition, where $< p >$ is contained in one vertex group, $< q >$ is contained in a second vertex group, and the number of specializations of the edge groups (up to the imaginaries that were presented in the previous section) is (uniformly) bounded for each equivalence class of $E(p,q)$.

The approach to such a bound, combines the techniques that were used in constructing the Duo envelope in theorem 1.4 (mainly the sieve procedure for quantifier elimination that was presented in [Se6]), together with the techniques that were used to construct formal (graded) Makanin-Razborov diagrams in sections 2-3 in [Se2], and the proof of the existence of a global bound on the number of rigid and strictly solid families of specializations of rigid and solid limit groups, that was presented in sections 1-2 in [Se3]. In the next section we show how to use the separation of variables that is obtained in this section to finally analyze definable equivalence relations.

Let $F_k =< a_1, \dots, a_k >$ be a non-abelian free group, and let $E(p,q)$ be a definable equivalence relation over F_k. Recall that with the definable equivalence relation, $E(p,q)$, being a definable set, one associates (using the sieve procedure) finitely many (terminal) rigid and solid limit groups, $Term_1, \dots, Term_s$, with each of the terminal limit groups $Term_i$ there are 4 sets associated, $B_j(Term_i)$, $j = 1, \dots, 4$, and that the definable set $E(p,q)$ is the set:

$$E(p,q) \ = \cup_{i=1}^{s} \left(B_1(Term_i) \setminus B_2(Term_i)\right) \cup \left(B_3(Term_i) \setminus B_4(Term_i)\right).$$

By theorems 1.3 and 1.4, with the given definable equivalence relation $E(p,q)$, being a definable set, we can associate a Diophantine and a Duo envelopes. Let $G_1, \dots, G_t$ be the Diophantine envelope of the given definable equivalence relation $E(p,q)$, and let $Duo_1, \dots, Duo_r$, be its Duo envelope.

Let Duo be one of the Duo limit groups, $Duo_1, \dots, Duo_r$. By definition (see definition 1.1), Duo can be represented as an amalgamated product: $Duo =< d_1, p > *_{<d_0,e_1>} < d_0, e_1, e_2 > *_{<d_0,e_2>} < d_2, q >$. By construction, in Duo there exists a

subgroup that demonstrates that generic elements $<p,q>$ in Duo are indeed in the equivalence relation $E(p,q)$. This subgroup that we denote: $<x,y,u,v,r,p,q,a>$, is generated by the subgroup $<p,q>$, together with elements x for rigid and strictly solid specializations of some of the terminal limit groups, $Term_1,\dots,Term_s$, that are associated by the sieve procedure with $E(p,q)$, elements y,u,v for rigid and strictly solid specializations of some of the terminal limit groups of the Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions that are associated with some of these terminal limit groups, and elements for specializations of primitive roots of the specializations of edge groups in the graded abelian decomposition of some of the terminal limit groups, $Term_1,\dots,Term_s$, and in the graded abelian decompositions of the terminal limit groups of some of the Extra PS resolutions that are associated with them (see the proof of theorem 1.3). The subgroup $<x,y,u,v,r,p,q>$, being a subgroup of Duo, inherits a graph of groups decomposition from the presentation of Duo as an amalgamated product. We denote the subgroup $<x,y,u,v,r,p,q,a>$ of Duo by Ipr.

Ipr, being a subgroup of Duo, inherits a graph of groups decomposition from its action on the Bass-Serre tree that is associated with the amalgamated product:

$$Duo = <d_1,p> *_{<d_0,e_1>} <d_0,e_1,e_2> *_{<d_0,e_2>} <d_2,q> .$$

Suppose that in this graph of groups decomposition of the subgroup, Ipr, there exists an edge with a trivial edge group. In that case Ipr admits a non-trivial free product, $Ipr = A * B$, and since $<p> << d_1,p>$ and $<q> << d_2,q>$, so $<p>$ and $<q>$ are either trivial or can be embedded into vertex groups in the graph of groups decomposition that is inherited by Ipr. Therefore, either:

(i) $<p,q>$ is a subgroup of A.

(ii) $<p>$ is a subgroup of A and $<q>$ is a subgroup of B.

If (i) holds, then the restrictions of the specializations of Duo to specializations of Ipr are not generated by rigid and strictly solid specializations of the terminal limit groups that are associated with Ipr (together with primitive roots of the specializations of edge groups), which contradicts the construction of Duo (see definition 1.1 and theorem 1.4).

To deal with case (ii), we present the following theorem, that associates finitely many rigid and solid limit groups (with respect to the parameter subgroup $<p,q>$) with the equivalence relation $E(p,q)$, so that each couple, $(p,q) \in E(p,q)$, can be proved to be in $E(p,q)$ by a rigid or a strictly solid homomorphism from one of these limit groups. Furthermore there exist at most finitely many equivalence classes of the equivalence relation $E(p,q)$, for which couples (p,q) that do not belong to these exceptional classes, can be proved to be in $E(p,q)$ using rigid or strictly solid homomorphisms of one of the finitely many associated rigid and solid limit groups (with respect to $<p,q>$), and these rigid and solid homomorphisms do not factor through a free product (of limit groups) as in case (ii). To prove the theorem we apply once again the sieve procedure [Se6], which was originally used for quantifier elimination, and was also the main tool in the construction of the Diophantine and Duo envelopes.

Theorem 3.1. *Let $F_k = <a_1,\dots,a_k>$ be a non-abelian free group, and let $E(p,q)$ be a definable equivalence relation over F_k. There exist finitely many rigid and solid*

limit groups (with respect to $<p,q>$) that we denote: $Ipr_1,\dots,Ipr_w$, so that:

(1) *for every couple $(p,q) \in E(p,q)$ there exists a rigid or a strictly solid homomorphism (with respect to $<p,q>$) $h: Ipr_i \to F_k$, for some index i, that contains a proof that $(p,q) \in E(p,q)$.*

(2) *there exist at most finitely many equivalence classes of $E(p,q)$, so that for every couple $(p,q) \in E(p,q)$ that does not belong to one of these finitely many classes, there exists a rigid or a strictly solid homomorphism, $h: Ipr_i \to F_k$, for some index i, that contains a proof that $(p,q) \in E(p,q)$, and so that h, and every strictly solid homomorphism in the strictly solid family of h, does not factor through a homomorphism ν onto a free product (of limit groups) $A * B$, in which $\nu(p) \in A$ and $\nu(q) \in B$.*

Proof: Recall that by the sieve procedure for quantifier elimination, with the equivalence relation, $E(p,q)$, there is a finite collection of associated rigid and solid terminal limit groups, $Term_1,\dots,Term_s$. With a couple $(p,q) \in E(p,q)$, there exists a homomorphism from a subgroup $<x,y,u,v,r,p,q,a> \to F_k$, where:

(a) the elements x are mapped to rigid and strictly solid specializations of some of the rigid and solid terminal limit groups, $Term_1,\dots,Term_s$.

(b) elements y,u,v that are mapped to rigid and strictly solid specializations of some of the terminal limit groups of the Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions that are associated with some of the terminal limit groups, $Term_1,\dots,Term_s$.

(c) elements r that are mapped to specializations of primitive roots of the specializations of edge groups in the graded abelian decomposition of some of the terminal limit groups, $Term_1,\dots,Term_s$, and in the graded abelian decompositions of the terminal limit groups of some of the Extra PS resolutions that are associated with them (see the proof of theorem 1.3).

We look at the collection of all the homomorphisms from a subgroup $<x,y,u,v,r,p,q,a> \to F_k$, that verify that a couple $(p,q) \in E(p,q)$. With this collection we can naturally associate (by section 5 in [Se1]) a finite collection of limit groups that we denote: $V_1,\dots,V_f$. With each of these limit groups we can associate its graded Makanin-Razborov diagram with respect to the parameter subgroup, $<p,q>$. We further look at the collection of rigid and strictly solid homomorphisms of rigid and solid limit groups in these diagrams, that verify that a couple $<p,q> \in E(p,q)$ (note that if a couple $(p,q) \in E(p,q)$, then there exists such a rigid or a strictly solid homomorphism). With this collection of rigid and solid homomorphisms (with respect to $<p,q>$), we can associate finitely many limit groups, that we denote, $L_1,\dots,L_g$.

At this point we collect the subcollection of this collection of homomorphisms (i.e., rigid and strictly solid homomorphisms with respect to $<p,q>$ that verify that $(p,q) \in E(p,q)$) that factor through a free product of the form $A * B$, where A and B are limit groups, so that $<p> < A$ and $<q> < B$. By the standard methods of section 5 in [Se1], with the subcollection of such rigid and strictly solid homomorphisms we can naturally associate a finite collection of limit groups (graded with respect to $<p,q>$), that we denote $M_1,\dots,M_e$.

By successively applying the shortening argument to the subcollection of homomorphisms that factor through a free product of limit groups (by considering the actions of the graded limit groups $M_1,\dots,M_e$ on the Bass-Serre trees corresponding to the free products $A * B$ through which they factor), we can replace this

subcollection of homomorphisms with a new subcollection, and the finite collection of limit groups, $M_1, \dots, M_e$, with a new finite subcollection, $GFD_1, \dots, GFD_d$, for which each of the (graded) limit groups, $GFD_1, \dots, GFD_d$, admits a free decomposition $A_j * B_j$, where $<p>< A_j$ and $<q>< B_j$.

Let $GFD_j = A_j * B_j$, so that $<p>< A_j$ and $<q>< B_j$. With A_j and B_j viewed as limit groups, we can naturally associate their taut Makanin-Razborov diagrams (see section 2 in [Se4] for the construction and properties of the taut diagram). With a taut resolution of A_j and a taut resolution of B_j, we naturally associate their free product which is a resolution of $GFD_j = A_j * B_j$. Let Res be such a resolution of GFD_j. Given the taut resolution Res of GFD_j we look at its collection of test sequences for which either:

(1) one of the rigid or strictly solid specializations that are specified by the specializations in the test sequence is not rigid or not strictly solid.
(2) the specializations of elements that are supposed to be mapped to primitive roots are divisible by one of the finitely many factors of the indices of the finite index subgroups that are associated with the rigid and solid limit groups $Term_1, \dots, Term_s$ and the Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions that are associated with them.
(3) there exist extra rigid or extra strictly solid specialization of one of the terminal limit groups $Term_1, \dots, Term_s$ or one of the auxiliary resolutions that are associated with them, and these extra specializations are not specified by the specializations of GFD_j that form the test sequence.

Using the construction of formal limit groups that appears in section 2 in [Se2], we associate with the collection of test sequences that satisfy one of the properties (1)-(3), a finite collection of closures of the resolutions Res, that we call Non-Rigid, Non-Solid, Root, and Extra resolutions (that satisfy properties (1), (2), and (3) in correspondence). Given an Extra resolution (property (3)), we further collect all its test sequences for which the extra rigid or strictly solid specialization (that was not specified by the corresponding specialization of GFD_j) is not rigid or not strictly solid. The collection of these test sequences can be also collected in finitely many closures of the resolution Extra, and we call these closures, Generic Collapse Extra resolutions.

Before we continue to the next step of the construction of the limit groups and resolutions that we'll use in order to prove theorem 3.1, we prove the following fairly straightforward lemma on the finiteness of equivalence classes that contain generic points in one (at least) of the resolutions Res that are associated with the various limit groups GFD_j.

Lemma 3.2. *Let GFD be one of the graded limit groups, $GFD_1, \dots, GFD_d$ constructed above, and let Res be one of its constructed taut resolutions. Then there exist at most finitely many equivalence classes of the definable equivalence relation $E(p,q)$, for which:*

(1) *for each of the finitely many equivalence classes there exist a test sequence $\{(z_n, x_n, y_n, u_n, v_n, r_n, p_n, q_n, a)\}_{n=1}^{\infty}$ of Res that restrict to couples $\{(p_n, q_n)\}_{n=1}^{\infty}$ that are in the equivalence class, and so that the specializations: $\{(x_n, y_n, u_n, v_n, r_n, p_n, q_n, a)\}_{n=1}^{\infty}$ form a proof that the couples $\{(p_n, q_n)\}_{n=1}^{\infty}$ are in the (definable set) equivalence relation $E(p,q)$.*

(2) *for each test sequence of* Res*,* $\{(z_n, x_n, y_n, u_n, v_n, r_n, p_n, q_n, a)\}_{n=1}^{\infty}$*, for which the restricted couples* $\{(p_n, q_n)\}_{n=1}^{\infty}$ *are in* $E(p,q)$*, and so that the specializations:* $\{(x_n, y_n, u_n, v_n, r_n, p_n, q_n, a)\}_{n=1}^{\infty}$ *form a proof that the couples* $\{(p_n, q_n)\}_{n=1}^{\infty}$ *are in the (definable set) equivalence relation* $E(p,q)$*, there exists an (infinite) subsequence of the test sequence that restrict to couples* $\{(p_n, q_n)\}_{n=1}^{\infty}$ *that are elements in one of the finitely many equivalence classes of* $E(p,q)$*.*

Proof: With each of the finitely many limit groups, $GFD_1, \dots, GFD_d$, we have associated finitely many (taut Makanin-Razborov) resolutions. Let Res be one of these finitely many resolutions. With Res we have associated a finite collection of Non-rigid, Non-solid, Left, Root, Extra PS, and Generic collapse Extra PS resolutions that are all closures of the resolution Res.

Since each of the limit groups, GFD_j, decomposes into a free product in which the subgroup $< p >$ is contained in one factor, and the subgroup $< q >$ is contained in a second factor, the resolution Res is composed from two distinct resolutions, Res_1 of a limit group that contains the subgroup $< p >$, and Res_2 of a limit group that contains the subgroup $< q >$.

The Non-rigid, Non-solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions that are associated with the resolution, Res, are all closures of Res. Every closure of the resolution, Res, is a free product of a closure of Res_1 and a closure of Res_2. Hence, with each of the resolutions that are associated with the resolution, Res, we can associate a closure of Res_1 and a closure of Res_2. Therefore, with each of the resolutions that are associated with Res, we can naturally associate cosets of some fixed finite index subgroups of the direct sums of the abelian vertex groups that appear along the abelian graph of groups decompositions that appear along the various levels of the resolutions, Res_1 and Res_2 (see definitions 1.15 and 1.16 in [Se2] for closures of a resolution, and for the coset of a finite index subgroup that is associated with a closure).

Suppose that there exists a test sequence of the resolution Res, so that:

(1) the specializations of the test sequence restrict to valid proofs that the associated couples, $\{(p_n, q_n)\}$, are in the definable set, $E(p,q)$.
(2) the specializations of the test sequence restrict to specializations of the abelian vertex groups in the abelian decompositions that are associated with the various levels of the resolutions Res_1 and Res_2, and these restrictions belong to fixed cosets of the finite index subgroups of the direct sums of these abelian vertex groups that are associated with Res_1 and Res_2, and the Non-rigid, Non-solid, Left, Root, Extra, and Generic Collapse Extra resolutions that are associated with the ambient resolution Res.

By the construction of the auxiliary resolutions, i.e., the Non-rigid, Non-solid, Left, Root, Extra and Generic Collapse Extra resolutions, that are associated with the resolution, Res, if for given coset of the finite index subgroups of the direct sums of the abelian groups that appear in the abelian decompositions that are associated with the various levels of Res_1 and Res_2, there exists a test sequence of Res that satisfy properties (1) and (2), then every test sequence of Res that satisfy (2) (with respect to the given cosets) satisfy part (1) as well.

Hence, given such cosets, there is a fixed equivalence class of $E(p,q)$ (that depends only on the given cosets), so that for all the test sequences of Res that satisfy (1) and (2) with respect to the two cosets, for large enough n, the couples,

$\{(p_n,q_n)\}$, belong to the equivalence class that is associated with the two cosets. Since there are finitely many resolutions, Res, that are associated with the limit groups, $GFD_1,\dots,GFD_d$, and with each resolution Res, there are only finitely many associated cosets, there are only finitely many equivalence classes of $E(p,q)$, for which there is a test sequence of one of the resolutions Res, that restrict to specializations, $\{(p_n,q_n)\}$, that belong to such equivalence class. This proves part (1) of the lemma for the (finite) collection of equivalence classes that are associated with the finitely many couples of cosets. Furthermore, from each test sequence of one of the resolutions, Res, that restrict to valid proofs that the specializations, $\{(p_n,q_n)\}$, are in the definable set $E(p,q)$, it is possible to extract a (test) subsequence that is associated with one of the couple of cosets that is associated with Res, and hence the test subsequence restricts to specializations, $\{(p_n,q_n)\}$,which for large enough index n are in the equivalence class that is associated with the two cosets that are associated with Res, which proves part (2) of the lemma.

□

The limit groups, $GFD_1,\dots,GFD_d$, collect all those couples, $(p,q)\in E(p,q)$, for which a proof that they are in $E(p,q)$, i.e., a rigid or strictly solid homomorphism: $h:<x,y,u,v,r,p,q,a>\to F_k$ that is associated with them (and satisfies the requirements from such a homomorphism to prove that $(p,q)\in E(p,q)$), factors through a free product $A*B$ in which A and B are limit groups, $<p><A$ and $<q><B$. Lemma 3.2 proves that there exist finitely many equivalence classes of $E(p,q)$, for which every test sequence of one of the resolutions in the taut Makanin-Razborov diagrams of the limit groups: $GFD_1,\dots,GFD_d$ that restrict to couples $\{(p_n,q_n)\}\in E(p,q)$ and to proofs that the couples are indeed in $E(p,q)$, can be divided into a finite set together with finitely many test sequences, so that each of the finitely many test sequences restricts to couples $\{(p_n,q_n)\}$ that belong to one of the finitely many equivalence classes of $E(p,q)$ that are associated (by lemma 3.2) with the limit groups $GFD_1,\dots,GFD_d$.

In order to prove theorem 3.1, we still need to study non-generic couples $(p,q)\in E(p,q)$ that can be extended to specializations of $GFD_1,\dots,GFD_d$, and these extended specializations are valid proofs that demonstrate that these non-generic couples $(p,q)\in E(p,q)$. To do that we need to construct new limit groups that admit homomorphisms that do not factor through a free product $A*B$, in which A and B are limit groups, $<p><A$ and $<q><B$, and verify that these non-generic couples, (p,q), are indeed in $E(p,q)$. to construct these new limit groups, we apply (once again) the sieve procedure [Se6], that was originally presented as part of the quantifier elimination procedure.

Let GFD be one of the limit groups $GFD_1,\dots,GFD_d$, and let Res be one of the resolutions in its taut Makanin-Razborov diagram. With Res we associate a new collection of Extra limit groups. Suppose that the limit group, GFD_j, that is associated with the resolution Res, admits a free product, $GFD_j=A_j*B_j$, and Res is composed from two taut resolutions, Res_1 of A_j (or a quotient of A_j), and Res_2 of B_j (or a quotient of B_j).

With the resolutions, Res, we associate two types of *Extra limit groups*, that we denote *Exlim*. First we look at all the specializations of (the completion of) Res_1 for which there exists a test sequence of specializations of (the completion of) Res_2, so that for each specialization in the combined sequence there exist extra rigid or (families of) strictly solid specializations (of one of the terminal limit groups $Term_1,\dots,Term_s$ or one of the terminal rigid or solid limit groups of the

Non-Rigid, Non-Solid, Left, Root, Extra PS, or Generic Collapse Extra PS resolutions that are associated with them) that are not specified by the corresponding specialization of the limit group GFD_j. Note that there is a global bound on the number of such (distinct) extra rigid or families of strictly solid specializations. By the techniques for constructing formal and graded formal limit groups (sections 2 and 3 in [Se2]), this collection of specializations can be collected in finitely many limit groups, and each has the form $Exlim_1 * Exlim_2$, where $Exlim_2$ is a closure of Res_2, and the completion of Res_1 is mapped into $Exlim_1$. Similarly, we look at the specializations of (the completion of) Res_2 for which there exists a test sequence of specializations of (the completion of) Res_1 so that the combined specializations have similar properties.

Note that with an extra limit group, $Exlim = Exlim_1 * Exlim_2$, we can naturally associate finitely many subgroups of $Exlim_1$ and $Exlim_2$, that are associated with the finite collection of extra rigid and extra solid specializations that are collected in the construction of $Exlim_1$ and $Exlim_2$, for which:

(1) each of these (finitely many) subgroups is a free product of two subgroups of $Exlim$. One is a subgroup of $Exlim_1$, and is rigid or solid with respect to $< p >$. The second is a subgroup of $Exlim_2$, and is rigid or solid with respect to $< q >$.
(2) each extra rigid or strictly solid specialization that is collected by $Exlim$ is a specialization of one of these subgroups of $Exlim$, which are a free product of rigid and solid subgroups of $Exlim_1$ and $Exlim_2$.

We continue with all the (finitely many) Extra resolutions and Extra limit groups of the prescribed structure, that were constructed from Res, i.e., from the couple of resolutions, Res_1 and Res_2. As in the quantifier elimination procedure (the sieve procedure), for each Extra resolution, and Extra limit group that are associated with Res (which is in particular a taut resolution), we collect all the specializations that factor and are taut with respect to the taut resolution, Res, and extend to specializations of either a resolution, $Extra$, or an Extra limit group, $Exlim$, and for which the elements that are supposed to be extra rigid or strictly solid specializations and are specified by these specializations collapse. This means that the elements that are supposed to be an extra rigid or strictly solid specializations are either not rigid or not strictly solid, or they coincide with a rigid specialization that is specified by the corresponding specialization of GFD, or they belong to a strictly solid family that is specified by GFD. These conditions on the elements that are supposed to be extra rigid or strictly solid specializations are clearly Diophantine conditions, hence, we can add elements that will demonstrate that the Diophantine conditions hold (see section 1 and 3 of [Se5] for more detailed explanation of these Diophantine conditions, and the way that they are imposed). By our standard methods (section 5 in [Se1]), with the entire collection of specializations that factor through an Extra resolution or an Extra limit group, and restrict to elements that are taut with respect to the (taut) resolution, Res, and for which the elements that are supposed to be extra rigid or strictly solid specialization satisfy one of the finitely many possible (collapse) Diophantine conditions, together with specializations of elements that demonstrate the fulfillment of these Diophantine conditions, we can associate finitely many limit groups. We denote these limit groups, $Collape_1, \dots, Collapse_f$, and call them $Collapse$ limit groups.

Let $Collapse$ be one of the Collapse limit groups, $Collapse_1, \dots, Collapse_f$.

With $Collapse$ we associate its graded Makanin-Razborov diagram with respect to the parameter subgroup $< p, q >$. We continue with all the rigid and strictly solid homomorphisms of rigid and solid limit groups in this Makanin-Razborov diagram. We look at all the rigid and strictly solid specializations of rigid and solid limit groups in this diagram, so that their restrictions to specializations of the corresponding limit group GFD are valid proofs that $(p, q) \in E(p, q)$, and for which the specializations factor through a free product $A * B$, so that A and B are limit groups, $< p >< A$ and $< q >< B$. With this collection of (rigid and strictly solid) homomorphisms we can associate a finite collection of limit groups (by the standard techniques that are presented in section 5 of [Se1]), that we denote, $R_1, \ldots, R_m$.

Given a limit group R_j, we can associate with it a graded Makanin-Razborov diagram in which every graded resolution (with respect to the parameter subgroup $< p, q >$) terminates in a rigid or in a solid limit group, and this terminal rigid or solid limit group admits a free decomposition $A * B$, in which A and B are limit groups, $< p >< A$ and $< q >< B$.

At this point we combine the graded Makanin-Razborov diagram of $Collapse$ with the graded Makanin-Razborov diagrams of each of the limit groups $R_1, \ldots, R_m$. Each of the resolutions in the graded Makanin-Razborov diagram of $Collapse$ terminates in a rigid or a solid limit group. We replace this graded resolution with finitely many resolutions. First we replace its terminating rigid or solid limit group by each of the quotients that are associated with it from the set $R_1, \ldots, R_m$. We continue each of the obtained resolutions (after performing the replacements) with the graded Makanin-Razborov diagram of the corresponding limit group R_j. By construction, each of the constructed resolutions starts with $Collapse$ and terminates with a limit group that admits a free product in which $< p >$ is a subgroup of one factor and $< q >$ is a subgroup of the second factor.
Given the obtained (graded) diagram of the limit group $Collapse$, we replace it with a strict (graded) diagram, according to the finite iterative procedure that is presented in proposition 1.10 in [Se2]. Note that each resolution in the strict diagram starts with a quotient of $Collapse$ and terminates with a limit group that admits a free product in which $< p >$ is contained in one factor and $< q >$ is contained in the second factor.

Let $CRes_1$ be a (graded) resolution in the strict diagram that is associated with $Collapse$. Note that given a homomorphism $h : Collapse \to F_k$ that factors through $CRes_1$, and h restricts to a specialization of GFD which is a valid proof that $(p, q) \in E(p, q)$, the rigid vertex groups in the graded abelian decompositions that are associated with the rigid and strictly solid specializations in GFD, remain elliptic through the entire combined resolution $CRes_1$ (i.e., remain elliptic in both resolutions from which $CRes_1$ is constructed).

Suppose that a rigid vertex group or an edge group in the abelian decomposition that is associated with the specialization of the extra rigid or strictly solid specialization in $Collapse$ does not remain elliptic through $CRes_1$. Then for all the specializations of the Extra resolution, or the Extra limit group, that is associated with the taut resolution of the graded limit group GFD, Res, that can be extended to specializations of $Collapse$ that factor through $CRes_1$, the Diophantine condition that we imposed on the extra rigid or strictly solid specialization in the resolution $Extra$ or the extra limit group $Exlim$, can be rephrased as a Diophantine condition that factor through a free product $A_1 * B_1$ that extends the free product of the resolution Res, and in particular, it is a free product in which

$< p >< A_1$ and $< q >< B_1$.

Suppose that all the rigid vertex groups and edge groups in the abelian decomposition that is associated with the specialization of the extra rigid or strictly solid specialization in $Collapse$ remain elliptic through $CRes_1$. Then all these rigid vertex groups can be conjugated into the factors of the terminal limit group of $CRes_1$, and hence, the same conclusion holds, i.e., for the relevant specializations that factor through $Extra$, or through $Exlim$, the Diophantine condition that we imposed on the extra rigid or strictly solid specialization in the resolution $Extra$, or the extra limit group, $Exlim$, can be rephrased as a Diophantine condition that factor through a free product $A_1 * B_1$ that extends the free product of the resolution $Extra$, or the Extra limit group, $Exlim$, and in particular, it is a free product in which $< p >< A_1$ and $< q >< B_1$.

Therefore, we can replace the limit groups $R_1, \ldots, R_m$, by starting with the Extra resolutions and the Extra limit groups that are associated with the taut resolutions in the Makanin-Razborov diagrams of the limit groups $GFD_1, \ldots, GFD_d$, that do all admit a free product in which $< p >$ is contained in one factor and $< q >$ is contained in the second factor, and on these resolutions we impose Diophantine conditions that factor as similar free products, i.e., these Diophantine conditions are imposed on the two factors of the Extra resolutions independently.

Since both the Extra resolutions and Extra limit groups, and the Diophantine conditions that are imposed on them, admit a free product in which $< p >$ is contained in one factor, and $< q >$ is contained in the second factor, to analyze the set of specializations that factor through an extra resolution or an extra limit group, and are taut with respect to an original resolution, Res, of one of the limit groups, $GFD_1, \ldots, GFD_d$, and so that these (extra rigid and strictly solid) specializations extend to specializations that satisfy the Diophantine conditions that are imposed on them, we can use the analysis of such resolutions that was presented in the sieve procedure [Se6], and apply it (independently) to each of the two factors of such Extra resolutions and Extra limit group. Both the limit groups, and the resolutions that are obtained after applying this analysis admit a free product in which $< p >$ is contained in one factor and $< q >$ is contained in the second factor.

We continue iteratively as in the sieve procedure. At each step we start with the collection of Extra resolutions and Extra limit groups that were constructed in the previous step. We look at the collection of specializations that factor through and are taut with respect to these resolutions, that satisfy one of finitely many Diophantine conditions, and with this collection we associate (using section 5 in [Se1]) finitely many limit groups that we denote $Collapse$.

With each of the obtained limit groups $Collapse$ we associate its graded Makanin-Razborov diagram (with respect to the subgroup $< p, q >$). Given each of the rigid and solid limit groups in this diagram, we collect all the rigid or strictly solid homomorphisms of it that factor through a free product of limit groups in which $< p >$ is contained in one factor, and $< q >$ is contained in the second factor. We collect all these rigid and strictly solid homomorphisms in finitely many limit groups, and with each such limit group we associate a graded Makanin-Razborov diagram (with respect to $< p, q >$) that terminates in rigid and solid limit groups that admit free products in which $< p >$ is contained in one factor, and $< q >$ is contained in the second factor.

We further combine the graded Makanin-Razborov diagrams of each of the limit

groups $Collapse$ with the graded Makanin-Razborov diagrams of the limit groups that are associated with the rigid and solid limit groups in these diagrams. Given a combined diagram we replace it by a strict diagram using the iterative procedure that appears in proposition 1.10 in [Se2]. Each resolution in the strict diagram that is associated with a limit group $Collapse$, starts with a quotient of $Collapse$ and terminates in a rigid or a strictly solid homomorphism that admits a free product in which $< p >$ is contained in one factor and $< q >$ is contained in the second factor.

As we concluded in the first step of the iterative procedure, from the structure of the strict diagram it follows that the Diophantine condition that forces the extra rigid or strictly solid specializations in the resolutions $Extra$ or the Extra limit groups, $Exlim$, we started this step with, can be imposed separately on the two factors of the Extra resolution $Extra$ or the Extra limit group, $Exlim$, so that the collection of specializations that factor through the Extra resolution or Extra limit group and are taut with respect to the resolution, Res, and do satisfy the (collapsed) Diophantine condition, can be collected in finitely many limit groups, and each of these limit groups admit a free product in which $< p >$ is contained in one factor, and $< q >$ is contained in the second factor.

We continue by associating (taut) resolutions with these limit groups according to the construction that is used in the sieve procedure [Se6]. Given each of these taut resolutions we associate with it non-rigid, non-solid, Root, and Extra resolutions as we did in the sieve procedure (sections 1 and 3 in [Se5]).

By lemma 3.2 there are at most finitely many equivalence classes of the equivalence relation $E(p,q)$ for which a test sequence of one of the constructed (taut) resolutions restricts to valid proofs that the corresponding couples $\{(p_n, q_n\}$ are in the set $E(p,q)$. We further associate with the constructed resolution finitely many Extra limit groups (as we did in the first step of the iterative procedure). We continue iteratively, and by the termination of the sieve procedure [Se6], the iterative procedure terminates after finitely many steps. We set the (graded) limit groups, $Ipr_1, \ldots, Ipr_w$, to be the rigid and solid limit groups $L_1, \ldots, L_g$, that were constructed in the initial step of the procedure, together with the finite collection of rigid and solid limit groups that appear in the graded Makanin-Razborov diagram of the limit groups $Collapse$ that are constructed along the various steps of the sieve procedure.

By construction, for every $(p,q) \in E(p,q)$ there exists a rigid or a strictly solid homomorphism from one of the rigid or solid limit groups, $L_1, \ldots, L_g$, that restricts to a valid proof that $(p,q) \in E(p,q)$. By applying lemma 3.2 in the various steps of the iterative procedure, there exist at most finitely many equivalence classes of $E(p,q)$ so that if $(p,q) \in E(p,q)$, and (p,q) does not belong to one of these classes, then there exists a rigid or a strictly solid homomorphism from one of the limit groups, $Ipr_1, \ldots, Ipr_w$, that restricts to a valid proof that the couple (p,q) is in the set $E(p,q)$, and furthermore, this rigid homomorphism and every strictly solid homomorphism which is in the same strictly solid family of the strictly solid homomorphism does not factor through a free product of limit groups in which $< p >$ is contained in one factor, and $< q >$ is contained in the second factor. Hence, theorem 3.1 follows.

□

Theorem 3.1 associates with the given definable equivalence relation, $E(p,q)$,

finitely many rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$, so that apart from finitely many equivalence classes, for each couple, $(p,q)\in E(p,q)$, there exists a rigid or a strictly solid family of homomorphisms from at least one of the limit groups, $Ipr_1,\dots,Ipr_w$, to the coefficient group F_k, so that the rigid homomorphisms or the strictly solid homomorphisms from the given strictly solid family do not factor through a free product of limit groups, $A*B$, in which $<p><A$ and $<q><B$, and each of these homomorphisms restricts to a valid proof that $(p,q)\in E(p,q)$.

The rigid and solid limit groups $Ipr_1,\dots,Ipr_w$ and their rigid and strictly solid families of homomorphisms that do not factor through graded free products and restrict to valid proofs, are the starting point for our approach to associating (definable) parameters with the equivalence classes of the definable equivalence relation $E(p,q)$.

Recall that by theorems 1.3 and 1.4, with the given definable equivalence relation $E(p,q)$, being a definable set, we can associate a Diophantine and a Duo envelopes. We denoted by $G_1,\dots,G_t$ the Diophantine envelope of the given definable equivalence relation $E(p,q)$, and by $Duo_1,\dots,Duo_r$, its Duo envelope.

We continue by modifying the construction of the Duo envelope that is presented in theorem 1.4, and use the collection of homomorphisms from the limit groups, $Ipr_1,\dots,Ipr_w$, that do not factor through a free product in which (the image of) $<p>$ is contained in one factor, and (the image of) $<q>$ is contained in the second factor (see the proof of theorem 1.4).

Let $G_1,\dots,G_t$ be the Diophantine envelope of the definable equivalence relation, $E(p,q)$ (see theorem 1.3). We start with the graded completions $G_1,\dots,G_t$ in parallel. With each graded completion G_j, $1\leq j\leq t$, we first associate a finite collection of duo limit groups.

To construct these duo limit groups, we look at the entire collection of graded test sequences that factor through the given graded completion, G_j, for which the (restricted) sequence of specializations $\{p_n\}$ can be extended to specializations of one of the limit groups, $Ipr_1(f,p,q),\dots,Ipr_w(f,p,q)$, so that these specializations of the subgroups Ipr_s are:

(1) rigid or almost shortest in their strictly solid family (see definition 2.8 in [Se3] for an almost shortest specialization).
(2) the images of the subgroups Ipr_s do not factor through a free product in which the subgroup $<p>$ can be conjugated into one factor, and the subgroup $<q>$ can be conjugated into the second factor.
(3) in each such test sequence of G_j the specializations of the subgroup $<q>$ is fixed.

With this entire collection of graded test sequences, and their extensions to specializations of the limit groups $Ipr_1,\dots,Ipr_w$, we associate finitely many graded Makanin-Razborov diagrams, precisely as we did in constructing the formal graded Makanin-Razborov diagram in section 3 of [Se2]. As in the formal Makanin-Razborov diagram, each resolution in the diagrams we construct terminates with a (graded) closure of the given graded completion, G_j, we have started with, amalgamated with another group along its base (which is the terminal rigid or solid limit group of the graded completion G_j), and the abelian vertex groups that commute with non-trivial elements in the base).

We continue as in the proof of theorem 1.4. By construction, a completion of a resolution in one of the constructed graded diagrams is a duo limit group. We

take the completions of the resolutions that appear in the finitely many diagrams that are associated with the graded completion G_j, to be the preliminary (finite) collection of duo limit groups that are associated with G_j. We proceed by applying the sieve procedure to the constructed duo limit groups, precisely as we did in the construction of the duo envelope in proving theorem 1.4.

Finally, we set the Duo envelope of the definable equivalence relation $E(p,q)$, that we denote, $TDuo_1,\dots,TDuo_m$, to be those duo limit groups that are associated with the Diophantine envelope, $G_1,\dots,G_t$, for which there exists a duo family having a test sequence, so that all the specializations in the test sequence restrict to elements (p,q) in $E(p,q)$, and for which the associated specializations of the subgroup, Ipr_s, testify that indeed the elements (p,q) are in $E(p,q)$ (i.e., in particular, a "generic point" in $TDuo_i$ restricts to elements in $E(p,q)$, and the corresponding restrictions to (the image of) Ipr_s are valid proofs that).

Note that by construction, the collection of duo limit groups that we constructed, $Tduo_1,\dots,Tduo_m$, satisfy all the properties (1-5) that are listed in theorem 1.4. Hence, it is justified to call the subgroups, $Tduo_1,\dots,Tduo_m$ a duo envelope of $E(p,q)$ (as we did for the duo limit groups that are constructed in theorem 1.4).

Proposition 3.3. *The Duo limit groups, $Tduo_1,\dots,Tduo_m$, that form a duo envelope of $E(p,q)$, and the rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$, have the following properties:*

1. *With each of the Duo limit groups $Tduo_i$, there is an associated homomorphism from an associated graded completion, $G_{j(i)}$, which is one of the graded completions that form the Diophantine envelope of $E(p,q)$. Furthermore, the graded completion $G_{j(i)}$ has the same structure as one of the two graded completions that are associated with $Tduo_i$. In fact, the corresponding graded completion in $Tduo_i$ is a graded closure of $G_{j(i)}$, and $G_{j(i)}$ is mapped into this closure preserving the level structure.*
2. *With each Duo limit group $Tduo_i$, there is an associated homomorphism from one of the limit groups, $Ipr_1,\dots,Ipr_w$, into $Tduo_i$ that does not factor through a free product in which $<p>$ is contained in one factor and $<q>$ is contained in the second factor. We denote the image of this limit group in $Tduo_i$, $<f,p,q>$.*
3. *$Tduo_i$, being a Duo limit group, admits the amalgamated product: $Tduo_i=<d_1^i,p>*_{<d_0^i,e_1^i>}<d_0^i,e_1^i,e_2^i>*_{<d_0^i,e_2^i>}<d_2^i,q>$. If both subgroups $<p>$ and $<q>$ are non-trivial in $<f,p,q>$, then the subgroup $<f,p,q>$ intersects non-trivially some conjugates of the distinguished vertex group in $Tduo_i$, $<d_0^i,e_1^i,e_2^i>$.*

Proof: Parts (1) and (2) follow from the construction of the duo limit group, $Tduo_i$, that starts with one of the graded completions, $G_{j(i)}$, and continues by collecting all the test sequences of $G_{j(i)}$, for which their restrictions to the subgroup $<p>$ can be extended to specializations of one of the limit groups, $Ipr_1,\dots,Ipr_w$.

Let $<f,p,q>$ be the image of one of the rigid or solid limit groups, $Ipr_1,\dots,Ipr_w$, in $Tduo_i$. The subgroup, $<f,p,q>$, inherits a graph of groups decomposition from the amalgamation of the ambient group $Tduo_i$:

$$Tduo_i=<d_1^i,p>*_{<d_0^i,e_1^i>}<d_0^i,e_1^i,e_2^i>*_{<d_0^i,e_2^i>}<d_2^i,q>$$

If both subgroups $<p>$ and $<q>$ are non-trivial in $<f,p,q>$, and $<f,p,q>$ intersects trivially all the conjugates of the vertex group, $<d_0^i,e_1^i,e_2^i>$, the graph of groups decomposition that is inherited by $<f,p,q>$ collapses into a non-trivial free product of $<f,p,q>$ in which $<p>$ is contained in one factor and $<q>$ is contained in a second factor. However, the duo limit group $Tduo_i$ was constructed from specializations of one of the limit groups, $Ipr_1,\dots,Ipr_w$, that do not factor through a free product of limit groups in which $<p>$ is contained in one factor and $<q>$ is contained in a second factor, a contradiction. Hence, in case both subgroups $<p>$ and $<q>$ are non-trivial, $<f,p,q>$ intersects non-trivially some conjugate of the vertex group, $<d_0^i,e_1^i,e_2^i>$, and part (3) follows.

□

Part (3) of proposition 3.3 uses the fact that the homomorphisms from the rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$, that we use to verify that (generic) couples (p,q) in the Duo limit groups, $TDuo_1,\dots,TDuo_m$, are in the given definable equivalence relation $E(p,q)$, do not factor through a free product in which $<p>$ is contained in one factor and $<q>$ is contained in the second factor, to deduce that $<f,p,q>$ intersects non-trivially some conjugates of $<d_0^i,e_1^i,e_2^i>$. The analysis of the specializations of these intersections is a key in our approach to associating parameters with the families of equivalence classes of the equivalence relation, $E(p,q)$.

For presentation purposes, we first continue by assuming that the Duo limit groups, $Tduo_1,\dots,Tduo_m$, terminate in rigid limit groups, i.e., that the abelian decomposition that is associated with the limit group $<d_0^i>$ is the trivial (graded) decomposition. We further assume that the graded closures that are associated with the duo limit groups, $Tduo_1,\dots,Tduo_m$, do not contain abelian vertex groups in any of their levels. Hence, in particular the subgroup $<d_0^i,e_1^i,e_2^i>$ is simply $<d_0^i>$. In the sequel, we will further assume that the Duo (and uniformization) limit groups that are constructed from these Duo limit groups and are associated with them, terminate in rigid limit groups, and the graded closures that are associated with them do not contain abelian vertex groups in any of their levels as well. These assumptions will allow us to present our approach to separation of variables, and to associating parameters with the equivalence classes of $E(p,q)$, while omitting some technicalities. Later on we omit these assumptions, and generalize our approach to work in the presence of both rigid and solid terminal limit groups, and when abelian groups do appear as vertex groups in the abelian decompositions that are associated with the graded closures that are associated with the constructed duo limit groups.

Proposition 3.4. *Let $Tduo_i$ be one of the Duo limit groups, $Tduo_1,\dots,Tduo_m$, and suppose that $Tduo_i$ terminates in a rigid limit group, i.e., that the abelian decomposition that is associated with $<d_0^i>$ is trivial. Suppose further that the two graded completions that are associated with $Tduo_i$ contain no abelian vertex groups in any of their levels.*

$Tduo_i$ being a duo limit group, admits a presentation as an amalgamated product: $Tduo_i=<d_1^i,p>*_{<d_0^i>}<d_2^i,q>$. Suppose that the subgroups $<p>$ and $<q>$ are both non-trivial in $Tduo_i$. By proposition 3.3, the subgroup $<f,p,q>$, which is the image of one of the rigid or solid limit groups, $Ipr_1,\dots,Ipr_w$, in $Tduo_i$, intersects non-trivially some conjugates of the edge group $<d_0^i>$. Let $H_i^1,\dots,H_i^e$ be these (conjugacy classes of) intersection subgroups.

Let $G_{j(i)}$ be the graded completion from the Diophantine envelope, $G_1, \dots, G_t$, that is mapped into $Tduo_i$. $G_{j(i)}$, being a graded completion, has a distinguished vertex group, which is a subgroup of its base subgroup. Then there exists a global integer $b_i > 0$, so that for any specialization of the distinguished vertex group of $G_{j(i)}$, there are at most b_i rigid specializations of $< d_0^i >$ that extend to generic (i.e., restrictions of (duo) test sequences) rigid and strictly solid specializations of $< f, p, q >$, that form valid proofs that the pairs (p, q) are in $E(p, q)$. In particular, these specializations of $< d_0^i >$ restrict to at most b_i conjugacy classes of specializations of the subgroups, $H_i^1, \dots, H_i^e$.

Proof: Since the subgroup, $< d_0^i >$, of the duo limit group, $Tduo_i$, is assumed to be rigid, the proposition follows from the existence of a uniform bound on the number of rigid specializations of a rigid limit group with a fixed value of the defining parameters (i.e., a bound that does not depend on the specific value of the defining parameters) that was proved in theorem 2.5 in [Se3].

□

Proposition 3.4 proves that for a given specialization of the distinguished vertex group in $G_{j(i)}$, there are at most boundedly many conjugacy classes of specializations of the corresponding subgroups, $H_i^1, \dots, H_i^e$, that may be associated with it. However, given an equivalence class of $E(p, q)$ we can't, in general, associate with it only finitely many conjugacy classes of specializations of the subgroups $H_i^1, \dots, H_i^e$. Hence, to obtain only boundedly many specializations or conjugacy classes of specializations of some "prefered" groups of parameters that are associated with each equivalence class (and not only with a specialization of the distinguished vertex group in $G_{j(i)}$), we need to construct *uniformization limit groups*.

Let $Tduo$ be one of the Duo limit groups, $Tduo_1, \dots, Tduo_m$. We assume as in proposition 3.4, that $Tduo$ terminates in a rigid limit group (i.e., the subgroup $< d_0 >$ admits a trivial graded JSJ decomposition), and that the two graded completions that are associated with $Tduo$ contain no abelian vertex group in any of their levels. $Tduo$, being a Duo limit group (with no abelian vertex groups that appear along the levels of its two associated graded completions), admits the amalgamated product: $Tduo = < d_1, p > *_{< d_0 >} < d_2, q >$. By part (1) of proposition 3.3, with $Tduo$ there is an associated homomorphism from an associated graded completion, G_j, which is one of the graded completions in the Diophantine envelope of $E(p, q)$. By part (2) of proposition 3.3, with $Tduo$ there is also an associated homomorphism from one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$, into $Tduo$. We denote the image of this homomorphism in $Tduo$, $< f, p, q >$. Note that by proposition 3.3 if both subgroups $< p >$ and $< q >$ in $< f, p, q >$ are non-trivial, then the intersection of $< f, p, q >$ with some conjugates of $< d_0 >$ is non-trivial. We denote by $H^1, \dots, H^e$ these intersection subgroups.

Suppose that there exists an equivalence class of $E(p, q)$, for which there exists an infinite sequence of conjugacy classes of specializations of $H^1, \dots, H^e$ that can be extended to couples of test sequences of the two graded completions that are associated with $Tduo$, so that restrictions of generic elements in these test sequences, (f_n, p_n, q_n), prove that the couples $(p_n, q_n) \in E(p, q)$, these test sequences restrict to valid proofs that the couples $(p_n, q_0(n))$ and $(q_0(n), q_n)$ belong to $E(p, q)$ (recall that $q_0(n)$ is the restriction of the specializations $d_0(n)$ to the elements q_0), and furthermore these test sequences restrict to sequences of distinct couples, $\{(p_n, q_n)\}$.

Note that it is a corollary of the quantifier elimination procedure (and the uni-

form bounds on rigid and strictly solid specializations of rigid and solid limit groups obtained in [Se3]), that there is a global bound on the size of all finite equivalence classes of a definable equivalence relation. To analyze the sets of specializations of the subgroups $H^1, \dots, H^e$ that are associated with the same infinite equivalence classes of $E(p,q)$, we construct finitely many limit groups (that are all associated with $Tduo$), that we call *uniformization limit groups*. To construct these limit groups we look at the collection of all the sequences:

$$\{(d_1(n), p_n, d_0, d_2(n), q_n, \hat{f}, \hat{d}_0, a)\}$$

for which:

(1) $\{(d_1(n), p_n, d_0)\}$ is a test sequence of the first graded completion that is associated with $TDuo$, and $\{(d_2(n), q_n, d_0)\}$ is a test sequence of the second graded completion that is associated with $TDuo$. These sequences restrict to proofs that the couples (p_n, q_0) and (q_0, q_n) are in $E(p,q)$, and the couples, $\{(p_n, q_n)\}$, in such a sequence are distinct.

(2) the sequence $\{(d_1(n), p_n, d_0, d_2(n), q_n)\}$ restricts to a sequence of specializations: $\{(f_n, p_n, q_n, a)\}$, that are rigid or strictly solid specializations of the rigid or solid limit group Ipr that is associated with $Tduo$ (see part (2) of proposition 3.3). Furthermore, the sequence of specializations, $\{(f_n, p_n, q_n, a)\}$, restricts to proofs that the sequence $\{(p_n, q_n)\}$ are in the given equivalence relation $E(p,q)$. As in constructing the Duo limit groups, $Tduo$, we assume that the couples (p_n, q_n) do not belong to the finitely many equivalence classes that are specified in theorem 3.1.

(3) the elements $\{(\hat{f}, d_0, \hat{d}_0)\}$ restrict to rigid or strictly solid specializations of one of the rigid and solid limit groups $Ipr_1, \dots, Ipr_w$, that prove that the specialization $< d_0 >$ and the specialization $\hat{d}_0$ belong to the same equivalence class in $E(p,q)$.

Using the techniques of sections 2 and 3 in [Se2], we can associate with the above collection of sequences (for the entire collection of Duo limit groups $Tduo_1, \dots, Tduo_m$) a finite collection of Duo limit groups, $Dduo_1, \dots, Dduo_u$. By construction, the subgroup $< \hat{f}, d_0, \hat{d}_0 >$ form the distinguished vertex groups of the constructed Duo limit groups, and the two graded completions that are associated with each such Duo limit group has the same structure as those of the Duo limit group $Tduo$ from which they were constructed.

We continue with each of the distinguished vertex groups $< \hat{f}, d_0, \hat{d}_0 >$ of the Duo limit groups, $Dduo_1, \dots, Dduo_u$. We view each of the vertex groups, $< \hat{f}, d_0, \hat{d}_0 >$, as graded limit groups with respect to the parameter subgroups $< \hat{d}_0 >$, and associate with them their graded taut Makanin-Razborov diagrams. With each resolution in these diagrams we naturally associate its graded completion (see definition 1.12 in [Se2] for the completion of a well-structured resolution).

Given each graded completion that is associated with a limit group $< \hat{f}, d_0, \hat{d}_0 >$, we construct a new limit group that starts with the graded completion of a resolution of a subgroup $< \hat{f}, d_0, \hat{d}_0 >$ (so that the terminal limit group of this graded completion contains the subgroup $< \hat{d}_0 >$), and on top of this completion we amalgam the two graded completions that were associated with the associated subgroup $Dduo$. i.e., we get a Duo limit group that has the same structure as the associated Duo limit group $Dduo$, but the distinguished vertex in $Dduo$ is replaced with a

graded completion that terminates in a rigid or a solid limit group that contains $<\hat{d}_0>$ (which is the parameter subgroup of this terminal rigid or solid limit group). We denote the limit groups that are constructed in this way from the Duo limit groups, $Dduo_1, \ldots, Dduo_u$, $Cduo_1, \ldots, Cduo_v$.

The distinguished vertex group in the Duo limit groups $Dduo_1, \ldots, Dduo_u$, that was the limit group $< d_0, \hat{f}, \hat{d}_0 >$, was replaced by the completions of the resolutions in the taut graded Makanin-Razborov diagrams of these groups (with respect to $<\hat{d}_0>$) to obtain the Duo limit groups $Cduo_1, \ldots, Cduo_v$. Each of the obtained Duo limit groups, that we denote $Cduo$, has the structure of a completion (of a resolution of $< d_0, \hat{f}, \hat{d}_0 >$), and on top of this completion we amalgam two additional completions, which are the two completions that are associated with the Duo limit group, $Tduo$, from which it was constructed.

Since the Duo limit group, $Cduo$, is constructed from 3 completions, we can naturally associate generic points with it, i.e., test sequences that are composed from test sequence of the completion of the resolution of the limit group, $< d_0, \hat{f}, \hat{d}_0 >$, that is extended to be a test sequence of the two completions that are amalgamated with that completion and these have the structure of the two completions that are associated with the Duo limit group, $Tduo$, from which the limit group $Cduo$ was constructed.

By construction, there exist generic points of the Duo limit group $Tduo$, and its associated Duo limit group, $Dduo$, i.e., (double) test sequences of the two completions that are associated with each of them, that restrict to proofs that the couples $(p_n, q_0(n))$ and $(q_0(n), q(n))$ are in $E(p,q)$, and restrict to specializations $\{(f_n, p_n, q_n)\}$ of the associated limit group Ipr, that restrict to proofs that the couples $\{(p_n, q_n)\}$ are in the given definable set $E(p,q)$. However, there is no guarantee that there exists a generic point of the limit group $Cduo$, that is constructed from $Dduo$, with these property, i.e., that there exists a (triple) test sequence, that is composed from test sequences of the 3 completions that form the Duo limit group $Cduo$, so that this (triple) test sequence restricts to proofs that the couples $(p_n, q_0(n))$ and $(q_0(n), q_n)$ are in $E(p,q)$ and to specializations, $\{(f_n, p_n, q_n)\}$ and $\{(\hat{f}_n, d_0(n), \hat{d}_0(n))\}$, that restrict to proofs that the couples $\{(p_n, q_n)\}$ are in $E(p,q)$, and that for each n the couple $(d_0(n), \hat{d}_0(n))$ are in the same equivalence class of $E(p,q)$.

Therefore, we start with the Duo limit group $Cduo$, and apply the sieve procedure to it, in the same way that we constructed the Diophantine and Duo envelopes in theorems 1.3 and 1.4. First, we look at all the (triple) test sequences of $Cduo$ for which specializations of subgroups of $< f, p, q >$ and $< \hat{f}, p, q >$ and specializations of the the subgroup that is supposed to demonstrate that the specializations of (p, q_0) and (q_0, q) are in $E(p,q)$ that were supposed to be rigid or strictly solid do not have this property. With this collection of test sequences we associate Non-Rigid and Non-Solid closures of $Cduo$. Similarly we construct Root and Extra closures of Duo. Given each of the extra resolutions we associate with it a (canonical, finite) collection of Generic Collapse closures of $Cduo$ (which are closures of the Extra closures), and Collapse limit groups. Given a Collapse limit group, we analyze its (Duo) resolutions using the analysis of quotient resolutions, that is used in the sieve procedure [Se6]. However, when we analyze the resolutions of a Collapse limit group, we analyze only those resolutions that have a similar structure as that of $Cduo$, i.e., that are built from a completion to which two other completions

are amalgamated and these two completions are closures of the two completions that are amalgamated to the completion of a resolution of $<\hat{f},d_0,\hat{d}_0>$ in the construction of $Cduo$. By continuing this construction iteratively, according to the steps of the sieve procedure, we finally obtain a finite collection of Duo limit groups, that we denote $Sduo_1,\ldots,Sduo_h$. Each of these Duo limit groups is constructed from a completion to which we amalgamate two closures of the completions that are amalgamated in the construction of $Cduo$. The sieve procedure that was used to construct the Duo limit groups, $Sduo_1,\ldots,Sduo_h$, guarantees that they have the following properties.

Proposition 3.5. *The Duo limit groups, $Sduo_1,\ldots,Sduo_h$, that are associated with the Duo limit groups, $Cduo_1,\ldots,Cduo_v$, have the following properties:*

1. *Each of the Duo limit groups, $Sduo_i$, is constructed from a completion that we denote $B(Sduo_i)$, that contains the subgroup $<\hat{f},d_0,\hat{d}_0>$, to which we amalgamate two completions that are closures of the completions that are amalgamated to the base completion in the associated Duo limit group $Cduo$. We denote these two completions $Cl_p(Sduo_i)$ and $Cl_q(Sduo_i)$.*
2. *there is a homomorphism from the associated Duo limit group $Cduo$ into $Sduo_i$ that maps the base completion in $Cduo$ into $B(Sduo_i)$, and the two completions that are amalgamated to the base completion in $Cduo$ into their closures, $Cl_p(Sduo_i)$ and $Cl_q(Sduo_i)$, so that the map preserves the level structure of the two completions.*
3. *The base completion $B(Sduo_i)$ terminates in either a rigid or a solid limit group with respect to the subgroup, $<\hat{d}_0>$.*
4. *for each $Sduo_i$ there exists two maps from either one or two of the limit groups, $Ipr_1,\ldots,Ipr_w$, into $Sduo_i$. Their images are the subgroup $<f,p,q>$, and a subgroup of $<\hat{f},d_0,\hat{d}_0>$ that we denote $<\hat{f},q_0,\hat{q}_0>$.*

Proof: All the properties (1)-(4) follow in a straightforward way from the construction of the duo limit groups, $Sduo_1,\ldots,Sduo_h$ from the duo limit groups, $Cduo_1,\ldots,Cduo_v$.

□

As we did with the Duo limit groups $Tduo$, for presentation purposes we assume that the terminal limit groups of each of the Duo limit groups, $Sduo_1,\ldots,Sduo_h$ are rigid, and there are no abelian vertex groups that appear in any of the levels of the 3 graded completions from which each of the duo limit groups, $Sduo_1,\ldots,Sduo_h$, is constructed.

The construction of the Duo limit groups, $Sduo_1,\ldots,Sduo_h$ allows us to present the construction of uniformization limit groups, that are the main tool that we use in order to obtain separation of variables, with which we will eventually be able to construct the set of parameters that are associated with the equivalence classes that are defined by the given equivalence relation $E(p,q)$.

Definition 3.6. *Let $Sduo$ be one of the obtained limit groups, $Sduo_1,\ldots,Sduo_h$. $Sduo$ is by construction a Duo limit group, and it contains a base (graded) completion that contains the subgroup $<\hat{f},d_0,\hat{d}_0>$, and on top of that completion, there are two amalgamated graded closures, one containing the subgroup $<d_1,p>$ that we denote $Cl_p(Sduo)$, and the second containing the subgroup $<d_2,q>$ that we denote $Cl_q(Sduo)$. We further denote the completion obtained from the base completion in*

Sduo, $B(Sduo)$, *to which we amalgam the closure*, $Cl_p(Sduo)$, $GC_p(Sduo)$, *and the completion obtained from* $B(Sduo)$ *to which we amalgam the closure*, $Cl_q(Sduo)$, $GC_q(Sduo)$.

Starting with a limit group $Sduo$, *and its associated graded completion*, $GC_p(Sduo)$, *we can apply the construction of the Duo envelope, that is presented in theorem 1.4, and associate with* $GC_p(Sduo)$ *a finite collection of Duo limit groups, so that one of the graded completions that is associated with these Duo limit groups is a graded closure of* $GC_p(Sduo)$, *and the distinguished vertex groups of these Duo limit groups contain the distinguished vertex group of* $GC_p(Sduo)$, *and in particular contain the subgroup* $< \hat{d}_0 >$.

Since the graded completion we started the construction with, $GC_p(Sduo)$, *is contained in* $Sduo$, *and* $Sduo$ *is obtained from* $GC_p(Sduo)$ *by amalgamating to it the closure*, $Cl_q(Sduo)$, *each of the Duo limit groups that is obtained from* $GC_p(Sduo)$ *using the construction of the Duo envelope (theorem 1.4) can be extended to* $Sduo$ *itself. i.e., with* $Sduo$ *we associate finitely many limit groups, where each of these limit groups is obtained from* $Sduo$ *by amalgamating it with the second graded completion (the one that is associated with the subgroup* $< \tilde{q} >$*) of one of the Duo limit groups that are constructed from it. We call the obtained limit groups* $uniformization$ *limit groups, and denote their entire collection (i.e., all the limit groups of this form that are obtained from the various Duo limit groups* $Sduo$*)*, $Unif_1, \dots, Unif_d$.

Uniformization limit groups is the main tool that will serve us to obtain separation of variables, which will eventually enable us to find the class functions that we are aiming for, i.e., class functions that separate classes and associate a bounded set of elements with each equivalence class. As we did with the Duo limit groups $Tduo$, for presentation purposes we assume that the terminal limit groups of each of the Duo limit groups that are associated with the various completions, $GC_p(Sduo_i)$, are rigid (and not solid) with respect to the subgroup $< \hat{d}_0 >$, and that the graded completions that are associated with these duo limit groups contain no abelian vertex groups in any of their levels. Since uniformization limit groups were constructed from these Duo limit groups, this implies that the terminal limit group of all the uniformization limit groups are rigid as well, and the graded completions that are associated with the constructed uniformization limit groups contain no abelian vertex groups in any of their levels. Later on we generalize our arguments and omit these assumptions.

Before we use the uniformization limit groups to further constructions we list some of their basic properties that will assist us in the sequel. These have mainly to do with the various maps from the rigid and solid limit groups, $Ipr_1, \dots, Ipr_w$, into uniformization limit groups.

Proposition 3.7. *Let* $Unif$ *be one of the uniformization limit groups*, $Unif_1, \dots, Unif_d$. *Then:*

(1) *With* $Unif$ *there are 3 associated maps from the various rigid and solid limit groups*, $Ipr_1, \dots, Ipr_w$ *into* $Unif$. *Two of these maps are associated and are into the Duo limit group* $Sduo$ *from which* $Unif$ *was constructed. The images of these two maps are the subgroup* $< f, p, q >$, *and a subgroup of* $< \hat{f}, d_0, \hat{d}_0 >$ *that we denote* $< \hat{f}, q_0, \hat{q}_0 >$. *The third homomorphism is from one of the rigid or solid limit group*, $Ipr_1, \dots, Ipr_w$, *into the Duo*

limit group that was constructed from $GC_p(Sduo)$, and from which $Unif$ was constructed. We denote the image of this third map, $<\tilde{f},p,\tilde{q}>$. Furthermore, none of these 3 maps factor through a free product of limit groups in which $<p>$ is contained in one factor, and $<q>$ is contained in the second factor.

(2) *there exist generic points of $Unif$, i.e., sequences of specializations of $Unif$ that are composed from test sequences of the 4 completions from which $Unif$ is built, (Cl_p, Cl_q, $B(Sduo)$, and the completion that is associated with $<\tilde{q}>$ in the Duo limit group that is associated with $GC_p(Sduo)$ from which $Unif$ was constructed), for which the restrictions to specializations of the 3 subgroups, $<f,p,q>$, $<\hat{f},q_0,\hat{q}_0>$, and $<\tilde{f},p,\tilde{q}>$, restrict to proofs that the specializations of the couples (p,q),$(q_0,\hat{q}_0)$, and $(p,\tilde{q})$ are in the same equivalence class of $E(p,q)$. Furthermore, these test sequences restrict to proofs that the specializations of the couples (p,q_0) and (q_0,q) are in the same equivalence class of $E(p,q)$, and these test sequences restrict to distinct sequences of specializations $\{(p_n,q_n\}$.*

(3) *If the subgroups $<p>$ and $<q>$ in $Unif$ are non-trivial, then the subgroup $<\tilde{f},p,\tilde{q}>$ intersects non-trivially some conjugates of the terminal rigid vertex group in $Unif$. Let $\tilde{H}^1,\dots,\tilde{H}^c$ be the (conjugacy classes of) subgroups of intersection. Then there exists a global bound U, so that for every possible value of $\hat{d}_0$ for which there exists a test sequence with the properties that are described in part (2), there are at most U possible conjugacy classes of specializations of the subgroups $\tilde{H}^1,\dots,\tilde{H}^c$ that extend $\hat{d}_0$ and together are restrictions of conjugates of a rigid specialization of the terminal limit group of $Unif$.*

Proof: Parts (1) and (2) follow from the construction of the uniformization limit group, $Unif$, and from the construction of the duo limit group, $Sduo$, from which it was constructed. Part (3) follows from the uniform bound on the number of rigid specialization of a rigid limit group with the same value of the defining parameters, that was proved in theorem 2.5 in [Se3].

□

Uniformization limit groups are constructed as an amalgamation of a (Duo) limit group $Sduo$, and a Duo limit group that was associated with a graded completion, $GC_p(Sduo)$, that is contained in $Sduo$. This structure of uniformization limit groups enable them to reflect properties of generic points in fibers on one side (the $Sduo$ side) and universal properties on the other side.

Theorem 3.8. *Let $Unif$ be one of the uniformization limit groups that are associated with the given definable equivalence relation $E(p,q)$. Then:*

(i) *Let DE be the distinguished vertex group in $Tduo$, the Duo limit group from which $Sduo$, the Duo limit group that is associated with $Unif$, was constructed. Let $H^1,\dots,H^e$ be the (conjugacy classes) of intersections between the subgroup $<f,p,q>$ and conjugates of the distinguished vertex group of $Tduo$, DE. Recall that by part (3) of proposition 3.3, if both subgroups $<p>$ and $<q>$ are non-trivial, then these subgroups of intersection are non-trivial.*

Let $\hat{d}'_0$ be a specialization of $\hat{d}_0$ (i.e., a specialization of DE) that does not belong to the finitely many equivalence classes of $E(p,q)$ that are singled

out in theorem 3.1, for which there are infinitely many specializations of $H^1,\dots,H^e$ that are associated with the equivalence class of $\hat{d}'_0$, so that these specializations of $H^1,\dots,H^e$ can be extended to conjugates of rigid specializations of DE (the terminal limit group in $Tduo$), and these rigid specializations of DE can be extended to test sequences of $Tduo$ that restrict to valid proofs that the corresponding couples: (p_n,q_n), and (p_n,q_0), belong to the same equivalence class of $E(p,q)$, and each of these test sequences restrict to an infinite distinct sequence of couples, $\{(p_n,q_n)\}$. Then there exists a Duo limit group $Sduo_1$ that is constructed from $Tduo$, in which $H^1,\dots,H^e$ are not all contained in the distinguished vertex of $Sduo_1$.
If there are in addition infinitely many conjugacy classes of specializations of the subgroups $H^1,\dots,H^e$ with the same properties, then at least one of the subgroups $H^1,\dots,H^e$ is not contained in a conjugate of the distinguished vertex in $Sduo_1$.

(ii) *Let $\hat{d}'_0$ be a specialization of $\hat{d}_0$ that does not belong to the finitely many equivalence classes that are singled out in theorem 3.1, for which there are only finitely many conjugacy classes of specializations of $H^1,\dots,H^e$ that are associated with the equivalence class of $\hat{d}'_0$, so that these specializations of $H^1,\dots,H^e$ can be extended to conjugates of rigid specializations of DE (the terminal limit group in $Tduo$) that can be extended to test sequences of $Tduo$ that restrict to valid proofs that the corresponding couples: (p_n,q_n), and (p_n,q_0), belong to the same equivalence class of $E(p,q)$, and so that these test sequences restrict to sequences of distinct couples, $\{(p_n,q_n)\}$. Then there exists a global bound (that does not depend on $\hat{d}'_0$) on the possible values of the conjugacy classes of the subgroups $H^1,\dots,H^e$ that can extend such a specialization of the elements d_0 which is in the equivalence class of $\hat{d}'_0$.*

(iii) *Let $\hat{d}'_0$ be a specialization of the elements $\hat{d}_0$, so that $\hat{d}'_0$ restricts to specializations $\hat{q}'_0$, and $\hat{q}'_0$ does not belong to the finitely many equivalence classes of $E(p,q)$ that are excluded in theorem 3.1. Suppose that $\hat{d}'_0$ extends to a test sequence of $Sduo$ that restricts to proofs that the couples $(p_n,q_0(n))$, and $(q_0(n),\hat{q}'_0)$ belong to the same equivalence class in $E(p,q)$. Let $\tilde{q}'$ be a specialization of the elements $\tilde{q}$ that belongs to the equivalence class of $\hat{q}'_0$. Then $\tilde{q}'$ extends to a specialization of at least one of the uniformization limits groups $Unif$ that are constructed from $Sduo$, so that for this uniformization limit group $Unif$, $\tilde{q}'$ extends to a sequence of specializations, $\{(\tilde{f},p_n,\tilde{q}')\}$ that prove that the sequence of couples $\{(p_n,\tilde{q}')\}$ is in the given definable set $E(p,q)$, and the sequence $\{p_n\}$ is a test sequence for the completion, $GC_p(Sduo)$, that is contained in $Sduo$. Furthermore, the specializations of the corresponding test sequence of $GC_p(Sduo)$ restrict to proofs that the couples $\{(p_n,q_0(n))\}$, and $\{(q_0(n),\hat{q}'_0)\}$ are in the same equivalence class of $E(p,q)$.*

Proof: Part (i) follows from the constructions of the duo limit group, $Sduo$, and the uniformizations limit groups that are associated with it. Part (ii) follows from the construction of a uniformization limit group, and from the existence of a uniform bound on the number of rigid specializations of a rigid limit group, for any given value of the defining parameters (theorem 2.5 in [Se3]). Part (iii) follows since the uniformization limit groups that are associated with the duo limit group $Sduo$,

collect all the test sequences of $GC_p(Sduo)$ nd all the specializations of the subgroup $< q >$, so that the restriction of the test sequences of $GC_p(Sduo)$ to the subgroup $< p >$, and the values of the subgroup, $< q >$, can be extended to a sequence of specializations of one of the rigid or solid limit groups, $Ipr_1, \ldots, Ipr_w$. If $\tilde{q}'$ belong to the same equivalence class as $\hat{q}_0'$, such a test sequence clearly exists for a test sequence of $GC_p(Sduo)$ and $\tilde{q}'$, and part (iii) follows.

□

By construction, Uniformization limit groups admit 3 different maps from the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, into them. These maps prove that (generic specializations of) the couples, (p, q), $(q_0, \hat{q}_0)$, and $(p, \tilde{q})$, are in the definable set $E(p, q)$. The structure of the uniformization limit groups, and in particular their ability to use both generic points of their associated Duo limit group $Sduo$, and the universality property of the collection of specializations of the elements $\tilde{q}$ that is associated with the (finite) collection of uniformization limit groups that is associated with a Duo limit group $Sduo$, enable us to "compare" between two of these maps, those that verify that generic specializations of the couples (p, q) and $(p, \tilde{q})$ are in $E(p, q)$. This comparison is crucial in our approach to constructing the desired class functions from the given definable equivalence relation $E(p, q)$.

Recall that by part (3) of proposition 3.7, if the subgroups $< p >$ and $< q >$ are non-trivial, then the subgroup $< \tilde{f}, p, \tilde{q} >$ of a uniformization limit group $Unif$, intersects non-trivially conjugates of the distinguished vertex group in $Unif$, in (conjugacy classes of) the subgroups: $\tilde{H}^1, \ldots, \tilde{H}^c$.

Since the collection of duo limit groups, $Tduo_1, \ldots, Tduo_m$, collect all the possible extensions of test sequences of the graded completions, $G_1, \ldots, G_t$, that form the Diophantine envelope of $E(p, q)$, to rigid and almost shortest strictly solid specializations of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, with the two subgroups of a uniformization limit group, $Unif$, $< f, p, q >$ and $< \tilde{f}, p, q >$, we can naturally associate two (possibly identical) of the duo limit groups, $Tduo_1, \ldots, Tduo_m$. By construction with $< f, p, q >$ we can associate the duo limit group $Tduo$, With $\tilde{f}, p, q >$ it is possible to associate another (possibly the same) duo limit group from the collection, $Tduo_1, \ldots, Tduo_m$, that we denote, $\tilde{T}duo$.

Let $Tduo$ be one of the duo limit groups, $Tduo_1, \ldots, Tduo_m$, and let DE be the distinguished vertex in $Tduo$. By the construction of $Tduo$, there is an associated map from one of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, into it, that we denoted $< f, p, q >$. $Tduo$ is a duo limit group, and by our assumptions it terminates in a rigid limit group and the two graded completions that are associated with it contain no non-cyclic abelian vertex group in any of their levels. Hence, $Tduo$, can be presented as an amalgamated product: $Tduo =< d_1, p > *_{DE=<d_0>} < d_2, q >$. If both subgroups, $< p >$ and $< q >$, are non-trivial in $Tduo$, then the subgroup $< f, p, q >$ intersects non-trivially some conjugates of the distinguished vertex group in $Tduo$. We denoted by $H^1, \ldots, H^e$ the conjugacy classes of these intersections.

Let $\hat{d}_0'$ be a specialization of d_0 (i.e., a specialization of a fixed generating set of DE) that does not belong to the finitely many equivalence classes of $E(p, q)$ that are singled out in theorem 3.1, for which there are infinitely many conjugacy classes of specializations of the subgroups $H^1, \ldots, H^e$ that are associated with the equivalence class of $\hat{d}_0'$, so that these specializations of $H^1, \ldots, H^e$ can be extended to conjugates of rigid specializations of DE (the terminal limit group in $Tduo$) that can be extended to test sequences of $Tduo$ that restrict to valid proofs that

the corresponding couples: (p_n,q_n), and (p_n,q_0), belong to the same equivalence class of $E(p,q)$, and to distinct couples of specializations: $\{(p_n,q_n)\}$. Then by part (i) of theorem 3.8 there exists a Duo limit group $Sduo$, that is constructed from $Tduo$, in which not all the images of the subgroups $H^1,\ldots,H^e$ can be conjugated into the distinguished vertex in $Sduo$, and so that $\hat{d}_0'$ can be extended to a test sequence of $Sduo$ that restricts to proofs that the couples: (p_n,q_n), $(p_n,q_0(n))$, and $(q_0(n),\hat{q}_0')$ belong to the same equivalence class of $E(p,q)$, the corresponding specializations of the subgroups $H^1,\ldots,H^e$ belong to distinct conjugacy classes, and the corresponding couples of specializations $\{(p_n,q_n)\}$ are distinct.
By the construction of the uniformization limit groups $Unif$, since this last conclusion holds for $Sduo$, it holds for at least one of the uniformization limit groups, $Unif$, that are associated with it.

Let $Unif$ be such a uniformization limit group, i.e., a uniformization limit group in which not all the subgroups. $H^1,\ldots,H^e$, can be conjugated into the distinguished vertex group in $Unif$. There are two maps of the limit groups $Ipr_1,\ldots,Ipr_w$ into $Unif$, with images $<f,p,q>$ and $<\tilde{f},p,q>$, that are associated with two (possibly identical) duo limit groups, $Tduo$ and $\tilde{T}duo$.
Recall that the duo limit groups, $Tduo_1,\ldots,Tduo_m$, encode all the extensions of test sequences of the graded completions, $G_1,\ldots,G_t$, that form the Diophantine envelope of $E(p,q)$, to rigid and almost shortest strictly solid specializations of the rigid and solid limit groups, $Ipr_1,\ldots,Ipr_w$, that do not factor through a free product in which $<p>$ is contained in one factor and $<q>$ is contained in the second factor, so that these extended test sequences restrict to valid proofs that the sequences of couples, $\{(p_n,q_n)\}$, are in the set $E(p,q)$, and the couples, $\{(p_n,q_n)\}$, are distinct.

Suppose that the second map of the limit groups, $Ipr_1,\ldots,Ipr_w$, into $Unif$, the one with image $<\tilde{f},p,q>$, is associated with $Tduo$ as well (i.e., $\tilde{T}duo=Tduo$). Furthermore, suppose that the images of the subgroups $H^1,\ldots,H^e$ in $<f,p,q>$ and in $<\tilde{f},p,q>$ are conjugate. Then, by the construction of the uniformization limit group $Unif$, the two images of at least one of these subgroups can not be conjugated into the distinguished vertex group in $Unif$.

Suppose that the map from one of the limit groups, $Ipr_1,\ldots,Ipr_w$, into $Unif$, with image $<\tilde{f},p,q>$, is associated with a duo limit group $Tduo_i$ which is not $Tduo$, or that it is associated with $Tduo$, but the images of the subgroups $H^1,\ldots,H^e$ under the two maps from $Ipr_1,\ldots,Ipr_w$ to $Unif$, with images, $<f,p,q>$ and $<\tilde{f},p,q>$, are not conjugate in $Unif$.

By proposition 3.4, under our assumption that the terminal limit groups of the duo limit groups, $Tduo_1,\ldots,Tduo_m$ are rigid, for each specialization d_0' of a (finite) generating set d_0 of the distinguished vertex group $DE=<d_0>$ of $Tduo$, there are at most boundedly many possible conjugacy classes of specializations of the subgroups, $H^1,\ldots,H^e$, that can extend d_0', so that there exists a test sequence of the duo family that is associated with d_0', that can be extended to shortest rigid and strictly solid specializations of one of the limit groups, $Ipr_1,\ldots,Ipr_w$, that is associated with $Tduo$ and these specializations will have one of the given conjugacy classes of specializations of the subgroups, $H^1,\ldots,H^e$.

Therefore, if the subgroup $<\tilde{f},p,q>$ is not associated with $Tduo$, or it is associated with $Tduo$, but the images of the subgroups $H^1,\ldots,H^e$ in $<f,p,q>$ and $<\tilde{f},p,q>$ are not pairwise conjugate, two of boundedly many such maps

are already present, and later on we will be able to apply the pigeon hole principle, to argue that after boundedly many steps two maps with pairwise conjugate subgroups $H_i^1,\dots,H_i^{e_i}$ (that are associated with one of the duo limit groups, $Tduo_1,\dots,Tduo_m$) must be present. This will eventually guarantee the termination of an iterative procedure that we present, that will finally give us the parameters for the equivalence classes of the given equivalence relation $E(p,q)$.

Given the definable equivalence relation $E(p,q)$, we started its analysis with its Diophantine envelope, $G_1,\dots,G_t$, and Duo envelope, $Duo_1,\dots,Duo_r$ (theorems 1.3 and 1.4). We further associated with $E(p,q)$ the rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$, so that their rigid and strictly solid specializations (with respect to the parameter subgroup $<p,q>$) restrict to valid proofs that the couples (p,q) are in $E(p,q)$, and these specializations do not factor through a free product in which $<p>$ is contained in one factor and $<q>$ is contained in the second factor, for all but finitely many equivalence classes of $E(p,q)$ (theorem 3.1). Then we collected all the possible extensions of test sequences of the graded completions, $G_1,\dots,G_t$, that form the Diophantine envelope of $E(p,q)$, to rigid and almost shortest strictly solid specializations of the rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$, and these were collected by the (finite) collection of duo limit groups, $Tduo_1,\dots,Tduo_m$ (see propositions 3.3 and 3.4), that form a duo envelope of $E(p,q)$ as well.

Into each of the duo limit groups, $Tduo_1,\dots,Tduo_m$, there is an associated map of one of the rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$. We denoted the image of the map from one of the rigid or solid limit groups, $Ipr_1,\dots,Ipr_w$, into a duo limit group, $Tduo$, by $<f,p,q>$. If both subgroups $<p>$ and $<q>$ are non-trivial in $Tduo$, then by proposition 3.3 the subgroup $<f,p,q>$ intersects some conjugates of the distinguished vertex group in $Tduo$ non-trivially. We denoted by $H^1,\dots,H^e$ the conjugacy classes of these intersection subgroups. be non-trivial.
If the number of specializations of conjugacy classes of specializations of the subgroups, $H^1,\dots,H^e$ for a given equivalence class is finite then it is globally bounded (for all such equivalence classes). For the entire collection of equivalence classes for which the number of conjugacy classes of specializations of $H^1,\dots,H^e$ is infinite, we have associated with the duo limit groups, $Tduo_1,\dots,Tduo_m$, a finite collection of duo limit groups $Sduo_1,\dots,Sduo_h$.

With each of the duo limit groups, $Sduo$, we have associated a finite collection of uniformization limit groups, that we denoted, $Unif_1,\dots,Unif_d$. Each of these uniformization limit groups admits a second map from one of the rigid and solid limit groups, that we denote $<\tilde{f},p,q>$. By the universality of the collection of duo limit groups, $Tduo_1,\dots,Tduo_m$, with each of the subgroups, $<f,p,q>$ and $<\tilde{f},p,q>$, it is possible to associate one of these limit groups. By construction $Tduo$ is associated with $<f,p,q>$, and we denoted $\tilde{T}duo$, the duo limit group (from the collection $Tduo_1,\dots,Tduo_m$) that is associated with the subgroup $<\tilde{f},p,q>$ ($Tduo$ and $\tilde{T}duo$ may be the same duo limit group).
If both subgroups $<p>$ and $<\tilde{q}>$ in $Unif$ are non-trivial, then the subgroup $<\tilde{f},p,\tilde{q}>$ intersects some conjugates of the distinguished vertex group in $Unif$ non-trivially. We denote by $\tilde{H}^1,\dots,\tilde{H}^c$ the conjugacy classes of these subgroups of intersection.

To associate parameters with the various equivalence classes of the given equivalence relation $E(p,q)$, and obtain separation of variables, we repeat these constructions iteratively. The constructions that we perform in the second step of

the iterative procedure, depend on whether the two maps from the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, into a uniformization limit group $Unif$, with images $< f, p, q >$ and $< \tilde{f}, p, q >$, are associated with the same duo limit group $Tduo$ (i.e., if $\tilde{T}duo = Tduo$), and if so whether the images of the subgroups $H^1, \ldots, H^e$ under the two associated maps, are pairwise conjugate, or not.

Suppose that the two maps with images, $< f, p, q >$ and $< \tilde{f}, p, q >$, are associated with the same duo limit group $Tduo$, and the images of the subgroups, $< H^1, \ldots, H^e >$, under the two associated maps, are pairwise conjugate. Suppose that there exists an equivalence class of $E(p, q)$, which is not one of the finitely many equivalence classes that were singled out in theorem 3.1, for which there exist an infinite sequence of specializations of the subgroups, $\tilde{H}^1, \ldots, \tilde{H}^c$, that are not pairwise conjugate, that can be extended to test sequences of a uniformization limit group $Unif$ (i.e., sequences that restrict to test sequences of the 4 completions from which the uniformization limit group $Unif$ is composed), that restrict to pairwise non-conjugate specializations of the subgroups, $H^1, \ldots, H^e$, and so that the restrictions of these test sequences, $\{(f_n, p_n, q_n)\}$ and $\{(\tilde{f}_n, p_n, \tilde{q}_n)\}$, prove that the couples, $\{(p_n, q_n)\}$ and $\{(p_n, \tilde{q}_n)\}$, are in $E(p, q)$, and the restrictions $\{(\hat{f}_n, q_0(n), \hat{q}_0)\}$ prove that the couples $\{(q_0(n), \hat{q}_0)\}$ are in $E(p, q)$. Furthermore these test sequences restrict to valid proofs that the couples $\{(p_n, q_0(n))\}$ are in $E(p, q)$ (recall that $q_0(n)$ is the restriction of the specializations $d_0(n)$ to the elements q_0 and $\hat{q}_0$ is the restriction of the specialization $\hat{d}_0$), and the couples, $\{(p_n, q_n)\}$ and $\{(p_n, \tilde{q}_n)\}$, are distinct.

We collect all these equivalence classes and their associated specializations of the subgroup $\tilde{H}$ in a finite collection of uniformization limit groups, in a similar way to the construction of the uniformization limit groups, $Unif_1, \ldots, Unif_d$. With these we associate a finite collection of duo limit groups, in a similar way to the construction of the duo limit groups, $Cduo_1, \ldots, Cduo_v$. Then we apply the sieve procedure (that is presented in [Se6]), to associate with this collection of duo limit groups, a finite collection of duo limit groups, in a similar way to the construction of the duo limit groups, $Sduo_1, \ldots, Sduo_h$. With each of these duo limit groups we associate a finite collection of uniformization limit groups. We denote these uniformization limit groups, $Unif^2_1, \ldots, Unif^2_{d^2}$. For presentation purposes we continue to assume that the terminal limit groups of all the constructed duo limit groups are rigid (and not solid), and the graded completions that are associated with them contain no abelian vertex groups in any of their levels.

Let $Unif^2$ be one of the duo limit groups, $Unif^2_1, \ldots, Unif^2_{d^2}$, in which at least one of the subgroups, $\tilde{H}^1, \ldots, \tilde{H}^c$, can not be conjugated into the distinguished vertex group in $Unif^2$. By construction, there are 3 maps from the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, into $Unif^2$. Two of these maps with images, $< f, p, q >$ and $< \tilde{f}, p, \tilde{q} >$, are inherited from the associated uniformization limit group $Unif$, and are associated with the duo limit group $Tduo$ by our assumptions. The third map with image that we denote, $< f', p, q' >$, is also associated with one of the duo limit groups, $Tduo_1, \ldots, Tduo_m$. We denote the duo limit group with which $< f', p, q' >$ is associated, $Tduo'$.

By our assumptions the two subgroups, $< f, p, q >$ and $< \tilde{f}, p, \tilde{q} >$, are associated with the same duo limit group $Tduo$, and the images of the subgroups, $H^1, \ldots, H^e$, under these two maps are pairwise conjugate. Suppose that the third map from one of the subgroups, $Ipr_1, \ldots, Ipr_w$, into $Unif^2$ (with image $< f', p, q' >$), is

associated with $Tduo$ as well (i.e., $Tduo' = Tduo$), and the images of the subgroups, $H^1, \dots, H^e$, under the third map are pairwise conjugate to their images under the first two maps.

The uniformization limit group, $Unif^2$, was constructed from a uniformization limit group, $Unif$, and its associated duo limit group $Sduo$. Hence, by the universality of the collection of uniformization limit groups that are associated with the duo limit group, $Sduo$, with the third map from one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$, into $Unif^2$, with image $< f', p, q' >$, we can associate another (possibly the same) uniformization limit group that is associated with the duo limit group, $Sduo$, that we denote $Unif'$.

Suppose that the uniformization limit group, $Unif'$, that is associated with $Unif^2$, is $Unif$, the uniformization limit group from which $Unif^2$ was constructed. Suppose further, that the images of the subgroups, $\tilde{H}^1, \dots, \tilde{H}^c$, in the subgroup $< f', p, q' >$, are pairwise conjugate to the images of these subgroups in the subgroup, $< \tilde{f}, p, \tilde{q} >$.

If both subgroups, $< p >$ and $< q >$, are non-trivial, then the subgroup $< f', p, q' >$ of the uniformization limit group $Unif^2$, intersects non-trivially some conjugates of the distinguished vertex group in $Unif^2$. Let $H^{1'}, \dots, H^{b'}$ be conjugacy classes of these subgroups of intersection.

If the uniformization limit group that is associated with the subgroup $< f', p, q' >$ and the uniformization limit group $Unif^2$, is not $Unif$ (i.e., if $Unif'$ is not $Unif$), or if it is $Unif$, but the images of the subgroups, $\tilde{H}^1, \dots, \tilde{H}^c$, in the subgroup $< f', p, q' >$, are not pairwise conjugate to the images of these subgroups in the subgroup, $< \tilde{f}, p, \tilde{q} >$, then as we argued for the uniformization limit group $Unif^2$, the two maps with images $< \tilde{f}, p, \tilde{q} >$ and $< f', p, q' >$, occupies two of the boundedly many possibilities of such maps (where the bound is uniform and does not depend on the specific equivalence class of $E(p, q)$).

By our assumptions the two subgroups, $< f, p, q >$ and $< \tilde{f}, p, \tilde{q} >$, are associated with the same duo limit group $Tduo$, and the images of the subgroups, $H^1, \dots, H^e$, under these two maps are pairwise conjugate. If the third map from one of the subgroups, $Ipr_1, \dots, Ipr_w$, into $Unif^2$ (with image $< f', p, q' >$), is not associated with $Tduo$ as well (i.e., $Tduo'$ is not $Tduo$), or if the images of the subgroups, $H^1, \dots, H^e$, under the third map are not pairwise conjugate to their images under the first two maps, then by the same reasoning, the two maps with images $< f, p, q >$ and $< f', p, q' >$, occupies two of the boundedly many possibilities of such maps (where the bound is uniform and does not depend on the specific equivalence class of $E(p, q)$).

Suppose that the two maps with images, $< f, p, q >$ and $< \tilde{f}, p, \tilde{q} >$, are not associated with the same duo limit group $Tduo$, or that they are both associated with $Tduo$, and the images of the subgroups, $< H^1, \dots, H^e >$, under the two associated maps, are not pairwise conjugate. In this case, the second map, the one with image $< \tilde{f}, p, \tilde{q} >$, is associated with a duo limit group $\tilde{T}duo$. If both subgroups, $< p >$ and $< q >$, in $\tilde{T}duo$, are non-trivial, then the image of the map from one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$, in $\tilde{T}duo$, intersects non-trivially some conjugates of the distinguished vertex group in $\tilde{T}duo$. Let $\hat{H}^1, \dots, \hat{H}^a$ be the conjugacy classes of these subgroups of intersection.

Suppose that there exists an equivalence class of $E(p, q)$, which is not one of the finitely many equivalence classes that were singled out in theorem 3.1, for which

there exist an infinite sequence of specializations of the subgroups, $\hat{H}^1,\ldots,\hat{H}^a$, that are not pairwise conjugate, that can be extended to test sequences of a uniformization limit group $Unif$ (i.e., sequences that restrict to test sequences of the 4 completions from which the uniformization limit group $Unif$ is composed), that restrict to pairwise non-conjugate specializations of the subgroups, $H^1,\ldots,H^e$, and so that if both $<f,p,q>$ and $<\tilde{f},p,\tilde{q}>$ are associated with $Tduo$, then the specializations of the subgroups, $H^1,\ldots,H^e$, are not pairwise conjugate to those of $\hat{H}^1,\ldots,\hat{H}^{a=e}$, and the restrictions of these test sequences, $\{(f_n,p_n,q_n)\}$ and $\{(\tilde{f}_n,p_n,\tilde{q}_n)\}$, prove that the couples $\{(p_n,q_n)\}$ and $\{(p_n,\tilde{q}_n)\}$ are in $E(p,q)$, and the restrictions $\{(\hat{f}_n,q_0(n),\hat{q}_0)\}$ prove that the couples $\{(q_0(n),\hat{q}_0)\}$ are in $E(p,q)$. Furthermore these test sequences restrict to valid proofs that the couples $\{(p_n,q_0(n))\}$ are in $E(p,q)$ (recall that $q_0(n)$ is the restriction of the specializations $d_0(n)$ to the elements q_0 and $\hat{q}_0$ is the restriction of the specialization $\hat{d}_0$), and the couples $\{(p_n,q_n)\}$ and $\{(p_n,\tilde{q}_n)\}$ are distinct.

We collect all these equivalence classes and their associated specializations of the subgroups, $\hat{H}^1,\ldots,\hat{H}^a$, in a finite collection of duo limit groups, in a similar way to the construction of the duo limit groups, $Dduo_1,\ldots,Dduo_u$. With these we associate a finite collection of duo limit groups, in a similar way to the construction of the duo limit groups, $Cduo_1,\ldots,Cduo_v$. Then we apply the sieve procedure (that is presented in [Se6]), to associate with this collection of duo limit groups, a finite collection of duo limit groups, in a similar way to the construction of the duo limit groups, $Sduo_1,\ldots,Sduo_h$. With each of these duo limit groups we associate a finite collection of uniformization limit groups, that we denote (once again): $Unif^2_1,\ldots,Unif^2_{d^2}$. For presentation purposes we continue to assume that the terminal limit groups of all the constructed duo limit groups are rigid (and not solid), and the graded completions that are associated with them contain no abelian vertex groups in any of their levels.

Let $Unif^2$ be one of the uniformization limit groups, $Unif^2_1,\ldots,Unif^2_{d^2}$ in which at least one of the subgroups, $\hat{H}^1,\ldots,\hat{H}^a$, and at least one of the subgroups, $H^1,\ldots,H^e$, can not be conjugated into the distinguished vertex group in $Unif^2$. By construction, there are 3 maps from the rigid and solid limit groups, $Ipr_1,\ldots,Ipr_w$, into $Unif^2$. Two of these maps with images, $<f,p,q>$ and $<\tilde{f},p,\tilde{q}>$, are inherited from the associated uniformization limit group $Unif$, and are associated with the duo limit groups $Tduo$ and $\tilde{T}duo$ in correspondence, by our assumptions. The third map with image that we denote, $<f',p,q'>$, is associated with the construction of $Unif^2$, and is also associated with one of the duo limit groups, $Tduo_1,\ldots,Tduo_m$. We denote the duo limit group with which $<f',p,q'>$ is associated, $Tduo'$.

By our assumptions the two subgroups, $<f,p,q>$ and $<\tilde{f},p,\tilde{q}>$, are associated with the duo limit groups $Tduo$ and $\tilde{T}duo$ in correspondence, and if they are both associated with $Tduo$, then the images of the subgroups, $H^1,\ldots,H^e$, under these two maps are not pairwise conjugate. Suppose that the third map from one of the subgroups, $Ipr_1,\ldots,Ipr_w$, into $DPduo^2$ (with image $<f',p,q'>$), is associated with either $Tduo$ or $\tilde{T}duo$, and the images of the subgroups, $H^1,\ldots,H^e$, or the subgroups, $\hat{H}^1,\ldots,\hat{H}^a$, under the third map are pairwise conjugate to their images under the first or the second map in correspondence.

If both subgroups, $<p>$ and $<q>$, are non-trivial in $Unif^2$, then the subgroup $<f',p,q'>$ intersects some conjugates of the distinguished vertex group in

$Unif^2$ non-trivially. Let $H^{1'},\dots,H^{b'}$ be the conjugacy classes of these subgroups of intersection.

By our assumptions, the two subgroups, $< f,p,q >$ and $< \tilde{f},p,\tilde{q} >$, are associated with the duo limit groups $Tduo$ and $\tilde{T}duo$, and if $Tduo=\tilde{T}duo$ then the images of the subgroups, $H^1,\dots,H^e$, under these two maps are not pairwise conjugate. If the third map from one of the subgroups, $Ipr_1,\dots,Ipr_w$, into $Unif^2$ (with image $< f',p,q' >$), is not associated with $Tduo$ or $\tilde{T}duo$, or if the images of the subgroups, $H^1,\dots,H^e$, or $\hat{H}^1,\dots,\hat{H}^a$, under the third map are not pairwise conjugate to their images under the first or the second map, then three maps with images $< f,p,q >$, $< \tilde{f},p,\tilde{q} >$, and $< f',p,q' >$, occupies 3 of the boundedly many possibilities of such maps (where the bound is uniform and does not depend on the specific equivalence class of $E(p,q)$).

We continue iteratively. Suppose that there exists a uniformization limit group $Unif^2$, and an equivalence class of $E(p,q)$, which is not one of the finitely many classes that are singled out in theorem 3.1, for which there exist infinitely many conjugacy classes of specializations of the subgroups $H^{j'}$ that are associated with the image of the third map from $Ipr_1,\dots,Ipr_w$, into $Unif^2$ with image $< f',p,q' >$, that can be extended to test sequences of the duo limit group $Unif^2$ (i.e., sequences that restrict to test sequences of the 4 completions from which the duo limit group $DPduo^2$ is composed), so that restrictions of these test sequences to the subgroups, H^j and $\tilde{H}^j$ are pairwise non-conjugate, and the restrictions of the subgroups: $\{(f_n,p_n,q_n)\}$, $\{(\tilde{f}_n,p_n,\tilde{q}_n)\}$ and $\{(f',p,q')\}$, prove that the couples $\{(p_n,q_n)\}$, $\{(p_n,\tilde{q}_n)\}$ and $\{(p_n,q'_n)\}$, are in $E(p,q)$, and the restrictions of these test sequences prove that the couples, $\{(p_n,q_0(n)\}$, $\{(q_0(n),\hat{q}_0(n))\}$ and $\{(\hat{q}_0(n),q'_0)\}$ are in $E(p,q)$. Then we repeat these constructions, and obtain new uniformization limit groups, $Unif^3_1,\dots,Unif^3_{d^3}$ that admit 4 maps from the limit groups $Ipr_1,\dots,Ipr_w$ into each of them.

To obtain a set of parameters for the equivalence classes of the given definable equivalence relation, $E(p,q)$, we need to ensure a termination of this iterative procedure, that we'll leave us with only finitely many uniformization limit groups, $Unif^i$, and so that for each equivalence class (apart from the finitely many that are singled out in theorem 3.1) there will exist a uniformization limit group $Unif^i$ with only boundedly many possible conjugacy classes of values for the associated subgroups H^i (H^i are obtained as intersections between conjugates of the distinguished vertex group in $Unif^i$ and an associated image of one of the limit groups $Ipr_1,\dots,IPr_w$, $< f^i,p,q^i >$).

Theorem 3.9. *The iterative procedure for the construction of the uniformization limit groups, $Unif^i$, terminates after finitely many steps.*

Proof: Suppose that the iterative procedure does not terminate after finitely many steps. Since at each step finitely many uniformization limit groups are constructed, by Konig's lemma, if the procedure doesn't terminate there must exist an infinite path along it.

Each uniformization limit group along the infinite path is equipped with a map from one of the limit groups, $Ipr_1,\dots,Ipr_w$, into it (we denote this image $< f_j,p,q_j >$). Hence, by passing to a subsequence of the uniformization limit groups along the infinite path, we may assume that they are all equipped with a map from the same limit group, Ipr_i.

With each uniformization limit group from the chosen subsequence there is an associated map from the duo limit group Ipr_i into that uniformization limit group. By the construction of the uniformization limit groups, $Unif^j$, by passing to a further subsequence we may assume that the map from Ipr_i extends to a map from a fixed uniformization limit group, $Unif^{j_1}$. By passing iteratively to further subsequences we obtain maps from fixed uniformization limit groups, $Unif^{j_1}, Unif^{j_2}, \ldots$, into the uniformization limit groups from the corresponding subsequences.

Now, we look at the sequence of images, $< f_j, p, q_j >$, of the limit group, Ipr_i, in the uniformization limit groups, $Unif^j$, along the diagonal subsequence that is taken from the chosen subsequences of the infinite path. Each uniformization limit group in the diagonal subsequence is constructed as a limit of homomorphisms into a coefficient free group, F_k. With each uniformization limit group from the diagonal subsequence, (that we still denote) $Unif^j$, we associate a homomorphism, $h_j : Unif^j \to F_k$, that restricts and lifts to a homomorphism $s_j : Ipr_i \to F_k$. We choose the homomorphism h_j, so that it approximates the distances in the limit action of $Unif^j$ on the limit R^{n_j}-tree, for larger and larger (finite) subsets of elements in Ipr_i.

To analyze the sequence of homomorphisms $\{s_j\}$, and obtain a contradiction to the existence of an infinite path, we need the following theorem (theorem 1.3 in [Se3]), that gives a form of strong accessibility for limit groups.

Theorem 3.10 ([Se3],1.3). *Let G be a f.g. group, and let: $\{u_n \,|\, u_n : G \to F_k\}$ be a sequence of homomorphisms. Then there exist some integer $m \geq 1$, and a subsequence of the given sequence of homomorphisms, that converges into a free action of some limit quotient L of G on some R^m-tree.*

By theorem 3.10, from the sequence of homomorphisms, $s_j : Ipr_i \to F_k$, it is possible to extract a subsequence (that we still denote $\{s_j\}$) that converges into a free action of a limit quotient L of Ipr_i on some R^m-tree, for some integer $m \geq 1$. By construction, the homomorphism from Ipr_i into $Unif^j$ extends to homomorphisms from the uniformization limit groups, $Unif^{j_1}, Unif^{j_2}, \ldots$ into F_k. Hence, the limit action of the image of Ipr_i in $Unif^j$ on the associated R^{n_j}-tree contains at least j levels of infinitesimals. Since the homomorphisms $s_j : Ipr_i \to F_k$ were chosen to approximate these limit actions on larger and larger sets of elements of Ipr_i, it can not be that the limit action that is obtained from the sequence of homomorphisms $s_j : Ipr_i \to F_k$ according to theorem 3.10 contains only a finite sequence of infinitesimals. Therefore, we obtained a contradiction to the existence of an infinite path, and the procedure for the construction of uniformization limit groups terminate after finitely many steps.

$\square$

Theorem 3.9 asserts that the iterative procedure for the construction of the uniformization limit groups, $Unif^i$, terminates. Since the iterative procedure produces finitely many uniformization limit groups at each step, until its termination it constructs finitely many uniformization limit groups, $Unif^i$, that we denote $Unif_1, \ldots, Unif_v$ (we omit the notation for the step it was produced, since this will not be important in the sequel). With each uniformization limit group, $Unif_i$, there is an associated map from one of the rigid and solid limit groups $Ipr_1, \ldots, Ipr_w$ into $Unif_i$, that we denote, $< f_i, p, q_i >$. Note that by our assumptions all the terminal limit groups of the uniformization limit groups, $Unif_1, \ldots, Unif_v$, are rigid (and not solid). If the images of the subgroups $< p >$ and $< q >$ in $Unif_i$ are

both non-trivial, then the subgroup $< f_i, p, q_i >$ intersects some conjugates of the distinguished vertex group in $Unif_i$ non-trivially. We set $H_i^1, \ldots, H_i^{e_i}$ to be the conjugacy classes of these subgroups of intersection.

Theorem 3.11. *Suppose that all the uniformization limit groups, $Unif_1, \ldots, Unif_v$, and the duo limit groups that were used for their construction, terminate in rigid limit groups, and the graded completions that are associated with these groups contain no abelian vertex groups.*

Then for every equivalence class of $E(p,q)$, which is not one of the finitely many equivalence classes that are excluded in theorem 3.1, there exists a uniformization limit group, $Unif_i$, from the finite collection, $Unif_1, \ldots, Unif_v$, so that there exists a (positive) bounded number of conjugacy classes of specializations of the subgroups, $H_i^1, \ldots, H_i^{e_i}$, for which (cf. theorem 3.8):

1. *there exist specializations in the given conjugacy classes of specializations of the subgroups $H_i^1, \ldots, H_i^{e_i}$ that can be extended to rigid specializations of the distinguished (terminal) rigid vertex group in the uniformization limit group $Unif_i$, that can be further extended to test sequences of specializations that restrict to specializations of elements in the given equivalence class of $E(p,q)$.*

   The test sequences of specializations that extend the corresponding rigid specializations of the distinguished vertex group in the uniformization limit group $Unif_i$, restrict to valid proofs that the sequence of couples $\{(p_n, q_n)\}, \ldots, \{(p_n, q_n^i)\}$ are in the given equivalence class of $E(p,q)$, and to distinct sequence of couples $\{(p_n, q_n)\}, \ldots, \{(p_n, q_n^i)\}$. Furthermore, with the uniformization limit group, $Unif_i$, there are finitely many associated maps from the subgroups $Ipr_1, \ldots, Ipr_w$. With each such map, there are finitely many associated subgroups $H_{i,j}$ (that were associated with $Unif_i$ along the iterative procedure that constructs it). Then the test sequences that extend the rigid specializations of the distinguished vertex group in $Unif_i$, restricts to non pairwise conjugate specializations of the subgroups $H_{i,j}$ (for each level j), except for the bottom level subgroups, $H_i^1, \ldots, H_i^{e_i}$.
2. *the boundedly many conjugacy classes of specializations of the subgroups, $H_i^1, \ldots, H_i^{e_i}$, are the only conjugacy classes of specializations of these subgroups that satisfy part (1) for the given equivalence class of $E(p,q)$.*

Note that the bound on the number of conjugacy classes of specializations of the subgroups, $H_i^1, \ldots, H_i^{e_i}$, is uniform and it does not depend on the given equivalence class.

Proof: By the construction of the first level uniformization limit groups, $Unif_j^1$, for each equivalence class that is not one of the finitely many equivalence classes that are excluded in theorem 3.1, there exists a uniformization limit group, $Unif_j^1$, with (conjugacy classes of) specializations of the of the subgroups, $H_1, \ldots, H_{e_1}$, that satisfy part (1). If for a given equivalence class there are infinitely many such conjugacy classes, we pass the uniformization limit groups that were constructed in the second level. By continuing iteratively, and by the termination of the procedure for the construction of uniformization limit groups (theorem 3.9), for each given equivalence class (which was not excluded by theorem 3.1), we must reach a level in which there is a uniformization limit group with (conjugacy classes of) specializations of the subgroups, $H_1^i, \ldots, H_{e_i}^i$, that satisfy the conclusion of the

theorem.

□

Theorem 3.11 proves that for any equivalence class of $E(p,q)$ (except the finitely many equivalence classes that are singled out in theorem 3.1), there exists some uniformization limit group, $Unif_i$, which is one of the constructed uniformization limit groups, $Unif_1,\dots,Unif_v$, for which the subgroups, $H_i^1,\dots,H_i^{e_i}$, which are the conjugacy classes of intersecting subgroups between the subgroup, $< f_i,p,q_i >$, and conjugates of the distinguished vertex group in the uniformization limit group, $Unif_i$, admit only boundedly many conjugacy classes of specializations (that can be extended to test sequences of $Unif_i$ that satisfy part (1) in theorem 3.11).

Therefore, these boundedly many conjugacy classes of specializations of the subgroups, $H_i^1,\dots,H_i^{e_i}$, already enable us to construct a (definable) function from the collection of equivalence classes of $E(p,q)$ into a power set of the coefficient group F_k, so that the function maps each equivalence class of $E(p,q)$ into a (globally) bounded set. However, it is not guaranteed that the class function that one can define in that way, separates between different classes of $E(p,q)$.

The uniform bounds on the subgroups, $H_i^1,\dots,H_i^{e_i}$, does not yet give us the desired class function that we can associate with $E(p,q)$, i.e., a class function with "bounded" image for each equivalence class. It does give us a *separation of variables* that can be used as a step towards obtaining a desired class function. In order to obtain this separation of variables we need to look once again at the decomposition that we denote Λ_i, which is the decomposition that is inherited by the subgroup $< f_i,p,q_i >$ from the uniformization limit group, $Unif_i$, from which the uniformization limit group, $Unif_i$, was constructed.

Lemma 3.12. *With the notation of theorem 3.11, Λ_i, the graph of groups decomposition that is inherited by the subgroup, $< f_i,p,q_i >$, from the uniformization limit group, $Unif_i$, is either:*

(1) *Λ_i is a trivial graph, i.e., a graph that contains a single vertex. In that case either the subgroup $< p >$ or the subgroup $< q_i >$ is contained in the distinguished vertex in $Unif_i$, and in particular, it admits boundedly many values.*
(2) *Λ_i has two vertices, and (finitely many) edges between them. The subgroup $< p >$ is contained in one vertex group in Λ_i, and the subgroup $< q_i >$ is contained in the second vertex group in Λ_i. In that case the subgroups, $H_i^1,\dots,H_i^{e_i}$, contain conjugates of all the edge groups in Λ_i, except possibly a single edge, that connects the vertex that is stabilized by $< p >$ with the vertex that is stabilized by $< q_i >$ in Λ_i, that can have a trivial stabilizer.*

Proof: Since we assumed that the terminating limit groups of each of the uniformization limit groups, $Unif_i$, is rigid, each of the terminating limit groups, $Unif_i$, admits a splitting of the form, $Unif_i =< u,p > *_{<w>} < v,q_i >$, where $< w >$ is the terminal rigid limit group of the uniformization limit group. We will denote this splitting of $Unif_i$, Θ_i, and let T_i be the Bass-Serre tree that is associated with this splitting, Θ_i, of $Unif_i$.

The subgroup $< f_i,p,q_i >$ inherits a graph of groups decomposition, Λ_i, from the splitting, Θ_i. Since the subgroups $< p >$ and $< q_i >$ are elliptic in Θ_i, they can be both conjugated into vertex groups in Λ_i. Since the subgroups $< p >$ and $< q_i >$ stabilize adjacent vertices in the Bass-Serre tree, T_i, Bass-Serre theory for

actions of groups on simplicial trees, enables us to further assume that the vertex groups in Λ_i were chosen so that both $< p >$ and $< q_i >$ are contained in vertex groups in Λ_i.

First, suppose that the subgroups $< p >$ and $< q_i >$ fix the same vertex in the tree T_i. Since the uniformization limit group, $Unif_i$, is constructed from test sequences of specializations of the subgroup, $< u,p >$, and corresponding shortest possible specializations of the subgroup, $< f_i,p,q_i >$, the entire subgroup, $< f_i,p,q_i >$, must fix the same vertex in T_i. In that case either the subgroup $< p >$ or the subgroup $< q_i >$ is contained in the distinguished vertex in Θ_i, hence, it admits boundedly many values, and case (1) follows.

Suppose that the subgroup $< p >$ fixes a vertex in T_i, and the subgroup $< q_i >$ fixes a distinct vertex in T_i. Again since $Unif_i$ is constructed from test sequences of the subgroup, $< u,p >$, and corresponding shortest possible specializations of the subgroup, $< f_i,p,q_i >$, the action of the subgroup $< f_i,p,q_i >$ on a minimal invariant subtree of the tree T_i, contains only two orbits of vertices, one stabilized by conjugates of $< p >$ and the other by conjugates of $< q_i >$. Furthermore, in the graph of groups Λ_i, that is associated with this action, there are no loops that are based on these two vertices. Hence, Λ_i contains two vertices with edges between them. Since $< f_i,p,q_i >$ does not admit a free product in which the subgroup $< p,q_i >$ is contained in a factor, at most one of the edge groups in Λ_i can be trivial. All the other edge groups can be conjugated into the distinguished vertex group in the graph of groups Θ_i, hence, for each equivalence class, they values belong to boundedly many conjugacy classes.

$\square$

If we combine lemma 3.12 with theorem 3.11, for each equivalence class of $E(p,q)$ except the finitely many equivalence classes that are singled out in theorem 3.1, there exists a uniformization limit group, $Unif_i$, from the finite collection of the constructed uniformization limit groups, $Unif_1,\dots,Unif_v$, for which:

(1) with the uniformization limit group, $Unif_i$, there is an associated map from one of the rigid and solid limit groups, $Ipr_1,\dots,Ipr_w$, into $Unif_i$, with image, $< f_i,p,q_i >$.
(2) either one of the subgroups, $< p >$ or $< q_i >$ admits boundedly many values up to conjugacy, or the subgroup $< f_i,p,q_i >$ inherits a graph of groups decomposition, Λ_i, from the presentation of $Unif_i$ as an amalgamated product. Λ_i contains two vertices, where $< p >$ is contained in one vertex group and $< q_i >$ in the second vertex group.
(3) the edge groups in Λ_i are conjugates to some of the subgroups, $H_i^1,\dots,H_i^{e_i}$, and these groups admits only boundedly many conjugacy classes of specializations, that are associated with the given equivalence class, and satisfy the conditions that are presented in part (1) of theorem 3.11.

As we have already indicated, the bounded number of conjugacy classes of specializations of the subgroups, $H_i^1,\dots,H_i^{e_i}$, that are associated with a given equivalence class of $E(p,q)$, associated a bounded set with each equivalence class. However, these bounded sets may not separate between different equivalence classes. The graph of groups, Λ_i, that is inherited by the subgroup, $< f_i,p,q_i >$, from the uniformization limit group, $Unif_i$, which is either trivial (in which case one of the subgroups $< p >$ or $< q_i >$ admits only boundedly many values up to conjugacy), or it has two vertices, and (finitely many) edges between them, where the subgroup

$< p >$ is contained in one vertex group in Λ_i, and the subgroup $< q_i >$ is contained in the second vertex group in Λ_i, and the subgroups, $H_i^1, \ldots, H_i^{e_i}$, contain conjugates of the edge groups in Λ_i, can be viewed as a *separation of variables* (Λ_i separates between the subgroups $< p >$ and $< q_i >$). This separation of variables is the goal of this section, and the key for associating parameters with equivalence classes of the definable equivalence relation $E(p,q)$ in the next section, parameters that admit boundedly many values for each class, and these values separate between the different classes.

For presentation purposes, in all the constructions that were involved in obtaining the uniformization limit groups, $Unif_1, \ldots, Unif_v$, we assumed that the terminal limit group in all the associated duo limit groups are rigid (and not solid), and that there is no abelian vertex group in all the abelian decompositions that are associated with the various levels of the completions that are part of the constructed duo limit groups. Before we continue to the next section, and use the separation of variables we obtained to associate parameters with equivalence classes, we generalize the constructions we used, to omit these technical assumptions.

Suppose that the graded completions, $G_1, \ldots, G_t$, that form the Diophantine envelope of $E(p,q)$, contain no abelian vertex groups in any of the abelian decompositions that are associated with their various levels, and they terminate in either rigid or solid limit groups. The construction of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, and their properties that are listed in theorem 3.1, do not depend on the terminal limit groups of the graded completions, $G_1, \ldots, G_t$, being rigid or solid. Hence, we use them when some of the terminal limit groups of the the graded completions that form the Diophantine envelope are solid (and not only rigid).

From the graded completions, $G_1, \ldots, G_t$, their test sequences and their extensions to rigid or almost shortest specializations of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, we constructed the duo limit groups, $Tduo_1, \ldots, Tduo_m$. One of the graded completions that is associated with each of the duo limit groups, $Tduo_1, \ldots, Tduo_m$, has the same structure as the associated graded completion from the Diophantine envelope. For presentation purposes we further assume that the two graded completions that are associated with each of the duo limit groups, $Tduo_1, \ldots, Tduo_m$ contain no abelian vertex groups in any of their levels. We make no assumptions on the terminal limit groups of the duo limit groups, $Tduo_1, \ldots, Tduo_m$, i.e., they may terminate in either rigid or solid limit groups. Recall that with each of the duo limit groups, $Tduo_1, \ldots, Tduo_m$ there is an associated subgroup, $< f, p, q >$, that denotes the image of an associated map from one of the limit groups, $Ipr_1, \ldots, Ipr_w$, into it.

In propositions 3.3 and 3.4 we indicated the structure of the graph of groups decompositions that the subgroups $< f, p, q >$ inherit from the ambient duo limit groups, $Tduo_1, \ldots, Tduo_m$, in case these duo limit groups terminate in rigid limit groups and their associated completions contain no abelian vertex groups. The following proposition generalizes propositions 3.3 and 3.4 in case the terminal limit groups of $Tduo_1, \ldots, Tduo_m$ may be solid.

Proposition 3.13. *Let $Tduo_i$ be one of the Duo limit groups, $Tduo_1, \ldots, Tduo_m$, and suppose that the two graded completions that are associated with $Tduo_i$ contain no abelian vertex groups in any of their levels.*

Let the subgroup $< f, p, q >$, be the image in $Tduo_i$ of one of the rigid or solid

limit groups, $Ipr_1, \dots, Ipr_w$. If both subgroups $< p >$ and $< q >$ are non-trivial in $Tduo_i$, then $< f, p, q >$ intersects non-trivially some conjugates of the rigid vertex groups in the abelian decomposition that is associated with the terminal vertex group in $Tduo_i$. As in proposition 3.3, we denote by $H_i^1, \dots, H_i^e$ the conjugacy classes of these intersection subgroups.

Let $G_{j(i)}$ be the graded completion (from the Diophantine envelope of $E(p,q)$) that is associated (and mapped into) $Tduo_i$. Then there exists a global integer $b_i > 0$, so that for any rigid or strictly solid family of specializations of the (rigid or solid) distinguished vertex group in $G_{j(i)}$, there are at most b_i rigid or strictly solid families of specializations of the terminal rigid or solid terminal limit groups of $Tduo_i$ that extend the given family of specializations of the distinguished vertex group in the terminal limit group of $G_{j(i)}$, and so that these rigid or strictly solid families can be extended to generic rigid and strictly solid specializations of $< f, p, q >$. In particular, these specializations of the terminal limit group of $Tduo_i$ restrict to at most b_i conjugacy classes of specializations of the subgroups, $H_i^1, \dots, H_i^e$.

Proof: The proof is identical to the proof of proposition 3.4. Since the subgroup, $< d_0^i >$, of the duo limit group, $Tduo_i$, is assumed to solid, the proposition follows from the existence of a uniform bound on the number of strictly solid families of specializations of a solid limit group with a fixed value of the defining parameters (i.e., a bound that does not depend on the specific value of the defining parameters) that was proved in theorem 2.9 in [Se3].

□

Suppose that there exists a duo limit group $Tduo$, and an equivalence class of $E(p,q)$, for which there exists an infinite sequence of conjugacy classes of specializations of $H^1, \dots, H^e$ that can be extended to couples of test sequences of the two graded completions that are associated with $Tduo$, so that restrictions of generic elements in these test sequences, (f_n, p_n, q_n), prove that the couples $(p_n, q_n) \in E(p,q)$, these test sequences restrict to valid proofs that the couples $(p_n, q_0(n))$ and $(q_0(n), q_n)$ belong to $E(p,q)$ (recall that $q_0(n)$ is the restriction of the specializations $d_0(n)$ to the elements q_0), and furthermore these test sequences restrict to sequences of distinct couples, $\{(p_n, q_n)\}$.

In that case we can associate with the entire collection of such equivalence classes of $E(p,q)$, a finite collection of duo limit groups, $Sduo_1, \dots, Sduo_h$, precisely as we did in case the terminal limit groups of the duo limit groups, $TDuo_1, \dots, TDuo_m$ were rigid.

We further assume that the abelian decompositions that are associated with the various levels of the duo limit groups, $Sduo_1, \dots, Sduo_h$, contain no abelian vertex groups. In that case we can associate finitely many uniformization limit groups with each of the duo limit groups, $Sduo_1, \dots, Sduo_h$, precisely as we did in case the terminal limit groups of $Sduo_1, \dots, Sduo_h$ are all rigid. We Assume that these uniformization limit groups contain no abelian vertex groups in the abelian decompositions that are associated with their various levels. With each uniformization limit group, $Unif$, there is an associated duo limit group, $Sduo$.

As we did in the case the terminal limit groups of the uniformization limit groups, $Unif$, were all rigid, the image, $< \tilde{f}, p, \tilde{q} >$, of the map from one of the rigid and solid limit groups, $Ipr_1, \dots, Ipr_w$, into the uniformization limit group, $Unif$, intersects non-trivially some conjugates of the rigid vertex groups in the abelian decomposition that is associated with the terminal limit group of $Unif$, in

case the images of both subgroups $< p >$ and $< q >$ in the uniformization limit group, $Unif$, are non-trivial.

If there exist equivalence classes of $E(p,q)$ for which these (finitely many) intersection subgroups admit infinitely many conjugacy classes, we continue our constructions iteratively. In each step we construct new uniformization limit groups, $Unif^i$. If the image of both subgroups, $< p >$ and $< q >$, in $Unif^i$, are non-trivial, then the image of the map from one of the rigid and solid limit groups, $Ipr_1, \dots, Ipr_w$, into $Unif^i$ (that we denoted $< f_i, p, q_i >$), intersects non-trivially some conjugates of the rigid vertex groups in the abelian decomposition that is associated with the terminal limit group of $Unif^i$.

If there are equivalence classes of $E(p,q)$ for which these intersection subgroups admit infinitely many conjugacy classes of specializations we continue the next step. By the same argument that was used in proving theorem 3.9, the iterative process terminates after finitely many steps. After it terminates we are left with finitely many uniformization limit groups that were constructed in its various steps. For each equivalence class which is not one of the finitely many that were excluded in theorem 3.1, there exists some uniformization limit group, $Unif^i$, so that the intersection subgroups between the image of one of the rigid and solid limit groups, $Ipr_1, \dots, Ipr_w$, in $Unif^i$, and conjugates of the rigid vertex groups in the abelian decomposition that is associated with the terminal limit group of $Unif^i$, admit boundedly many values (up to conjugation) that are associated with the given equivalence class.

So far we explained how to generalize the outcome of the iterative procedure, or theorem 3.10, in case the terminal limit groups of the constructed duo and uniformization limit groups may be solid, but the graded completions that are associated with these groups contain no abelian vertex groups in any of their levels. To obtain from this conclusion a *separation of variables*, we still need to generalize lemma 3.12 in this case.

Lemma 3.14. *Suppose that the terminal limit groups in all the duo and uniformization limit groups that were constructed along our iterative procedure are rigid or solid, and the graded completions that are associated with them contain no abelian vertex groups in any of their levels. Let Λ_i be the graph of groups decomposition that is inherited by the subgroup, $< f_i, p, q_i >$, from the uniformization limit group, $Unif_i$. Then Λ_i is either:*

1. *Λ_i is a trivial graph, i.e., a graph that contains a single vertex. In that case either the subgroup $< p >$ or the subgroup $< q_i >$ is contained in the distinguished vertex group in the abelian decomposition that is associated with the distinguished vertex group in $Unif_i$, i.e., the vertex group that contains the coefficient group. In particular, either the subgroup $< p >$ or $< q >$ admits boundedly many values.*
2. *Λ_i has more than one vertex group, and both subgroups $< p >$ and $< q_i >$ are contained in the same vertex group. Like in case (1), in that case either the subgroup $< p >$ or the subgroup $< q_i >$ is contained in a conjugate of the distinguished vertex group in the abelian decomposition that is associated with the distinguished vertex group in $Unif_i$, and in particular, it admits boundedly many values. Furthermore, the subgroups, $H_i^1, \dots, H_i^{e_i}$, contain conjugates of all the edge groups in Λ_i, hence, the edge groups in Λ_i admit*

boundedly many values up to conjugacy.

(3) *Λ_i has more than one vertex, $< p >$ is contained in the stabilizer of one vertex and $< q_i >$ is contained in the stabilizer of another vertex in Λ_i. The subgroups, $H_i^1, \ldots, H_i^{e_i}$, contain conjugates of all the edge groups in Λ_i, except perhaps for one edge group, that is associated with the edge between the vertex that is stabilized by $< p >$ and the vertex that is stabilized by $< q_i >$, that can be trivial.*

Proof: The proof is similar to the argument that was used in proving lemma 3.12. From the construction of uniformization limit groups, each of the terminating limit groups, $Unif_i$, admits a graph of groups decomposition, Θ_i. The graph, Θ_i, is obtained from the graded abelian JSJ decomposition of the terminal rigid or solid limit group of the uniformization limit group, $Unif_i$, That we denote, Δ_i, by adding two vertices, the first is stabilized by a subgroup, $< u, p >$, and the second is stabilized by a subgroup, $< v, q_i >$. To the two additional vertices, we further add edges that connect these vertices to some of the rigid vertices (i.e., vertices with associated rigid vertex groups) in the graded abelian JSJ decomposition, Δ_i.

All the edge groups in the obtained graph of groups, are subgroups of the rigid vertex groups in the abelian JSJ decomposition, Δ_i. Hence, for each equivalence class of the given equivalence relation, $E(p, q)$, with which the uniformization limit group, $Unif_i$, is associated, there are at most boundedly many conjugacy classes of specializations of each of the edge groups in Θ_i, as well as of each rigid vertex group in Δ_i (which is part of Θ_i). Let T_i be the Bass-Serre tree that is associated with the graph of groups, Θ_i.

The subgroup $< f_i, p, q_i >$ inherits a graph of groups decomposition, Λ_i, from the splitting, Θ_i. Since the subgroups $< p >$ and $< q_i >$ are elliptic in Θ_i, they can be both conjugated into vertex groups in Λ_i. Since the subgroups $< p >$ and $< q_i >$ stabilize adjacent vertices in the Bass-Serre tree, T_i, Bass-Serre theory for actions of groups on simplicial trees, enables us to further assume that the vertex groups in Λ_i were chosen so that both $< p >$ and $< q_i >$ are contained in vertex groups in Λ_i.

First, suppose that the subgroups $< p >$ and $< q_i >$ fix the same vertex in the tree T_i. Since the subgroup, $< f_i, p, q_i >$ admits no free decomposition in which the subgroups, $< p >$ and $< q_i >$, are contained in the same factor, all the edge groups of Λ_i must be non-trivial and be contained in conjugates of the subgroups, $H_i^1, \ldots, H_i^{c_i}$. Hence, there are at most boundedly many values (up to conjugacy) of the edge groups in Λ_i that are associated with each equivalence class of the given definable equivalence relation, $E(p, q)$. Furthermore, in case both $< p >$ and $< q_i >$ are contained in the same vertex group of Λ_i, either $< p >$ or $< q_i >$ is contained in the distinguished vertex group of Θ_i (which is the distinguished vertex group in Δ_i, hence, there are at most boundedly many values of either $< p >$ or $< q_i >$ that are associated with each equivalence class of $E(p, q)$ with which $Unif_i$ is associated.

Suppose that $< p >$ and $< q_i >$ are contained in distinct vertex groups in Λ_i. The edge groups of Λ_i are all contained in conjugates of the subgroups: $H_i^1, \ldots, H_i^{c_i}$ (that admit at most boundedly many values up to conjugacy for each equivalence class that is associated with $Unif_i$). Since the subgroup, $< f_i, p, q_i >$, admits no free decomposition in which the subgroups $< p >$ and $< q_i >$ are contained in the same factor, all the edge groups in Λ_i that do not connect between the vertices that are stabilized by $< p >$ and $< q_i >$ must be non-trivial. If more than one edge

group that connects between the vertex groups that contain $< p >$ and $< q_i >$ is trivial, then $< f_i, p, q_i >$ admits a free product in which $< p >$ is contained in one factor and $< q_i >$ is contained in a second factor, and that contradicts the construction of the uniformization limit group, $Unif_i$, as it was constructed from sequences of specializations of the subgroup, $< f_i, p, q_i >$, that do not factor through a free product in which $< p >$ and $< q_i >$ are contained in distinct factors. Hence, in this case part (3) follows.

$\square$

Lemma 3.14 together with the uniform bound on the number of conjugacy classes of the edge groups in the abelian decompositions Λ_i, can be viewed once again as a separation of variables, in case the terminal limit groups of all the duo and uniformization limit groups that are constructed through our iterative procedure may be rigid or solid, and the graded completions that are associated with these groups contain no abelian vertex group in any of their levels.

So far we have assumed that the graded completions that are associated with all the duo and uniformization limit groups that are constructed along our iterative procedure contain no abelian vertex groups in the abelian decompositions that are associated with their various levels. At this stage we drop this assumption, and consider general duo and uniformization limit groups.

The construction of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, and their properties that are listed in theorem 3.1, do not depend on the structure of the graded completions, $G_1, \ldots, G_t$, that form the Diophantine envelope of the given equivalence relation, $E(p, q)$. Hence, we can use them as in the special cases that were analyzed before.

From the graded completions, $G_1, \ldots, G_t$, their test sequences and sequences of (rigid and almost shortest) homomorphisms from the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, we constructed the duo limit groups, $Tduo_1, \ldots, Tduo_m$, that can also serve as the duo envelope of $E(p, q)$. By construction, one of the graded completions that is associated with each of the duo limit groups, $Tduo_1, \ldots, Tduo_m$, is a closure of the graded completion, G_j, from which it was constructed. With each of the duo limit groups, $Tduo_1, \ldots, Tduo_m$ there is an associated subgroup, $< f, p, q >$, that denotes the image of an associated map from one of the limit groups, $Ipr_1, \ldots, Ipr_w$, into it. Propositions 3.3 is stated for general duo limit groups $Tduo_i$. In order to generalize propositions 3.4 and 3.12 to general duo limit groups, $Tduo_1, \ldots, Tduo_m$, we need the following observations.

Let $Tduo_i$ be one of the Duo limit groups, $Tduo_1, \ldots, Tduo_m$, and let the subgroup $< f, p, q >$, be the image in $Tduo_i$ of one of the rigid or solid limit groups, $Ipr_1, \ldots, Ipr_w$. Suppose that both subgroups, $< p >$ and $< q >$, are non-trivial in $Tduo_i$. $Tduo_i$ being a duo limit group, admits an amalgamated product decomposition:

$$Tduo_i =< d_1^i, p > *_{< d_0^i, e_1^i >} < d_0^i, e_1^i, e_2^i > *_{< d_0^i, e_2^i >} < d_2^i, q > .$$

Furthermore, the distinguished vertex group, $< d_0^i, e_1^i, e_2^i >$, admits a graph of groups decomposition that we denote, Γ_D^i, that is obtained from the graph of groups, $\Gamma_{< d_0 >}^i$, that is associated with the terminal rigid or solid limit group, $< d_0^i >$, so that to each rigid vertex group in $\Gamma_{< d_0 >}^i$ one further connects several (possibly none) free abelian vertex groups, which are subgroups of the subgroup <

$e_1^i, e_2^i >$, along some free abelian edge groups. This graph of groups decomposition, Γ_D^i, of the distinguished vertex group, $< d_0^i, e_1^i, e_2^i >$, can be used to construct a graph of groups decomposition of the ambient group, $Tduo_i$, a graph of groups that we denote Δ_i, with the following properties:

(1) The subgroups $< p >$ and $< q >$ are subgroups of vertex groups in Δ_i.

(2) Each of the subgroups that is obtained from a rigid vertex group in $\Gamma^i_{<d_0>}$, by connecting it to new vertices with abelian vertex groups (that are subgroups of $< e_1^i, e_2^i >$) in Γ_D^i, is a vertex group in Δ_i. We call such a vertex group in Δ_i, an *abelian star* vertex group. Each such an abelian star vertex group may be connected by an edge to either the vertex group that is stabilized by $< p >$, or to the vertex group that is stabilized by $< q >$ or to both. We call each of these edge groups, an *abelian star* edge group. Each such edge group is the fundamental group of a graph of groups containing the rigid vertex group from $\Gamma^i_{<d_0>}$ that appears in the graph of groups of the adjacent abelian star vertex group, and abelian vertex groups that are connected to it, abelian vertex groups which are a partial set of those abelian vertex groups that appear in the graph of groups that is associated with the adjacent abelian star vertex group.

All the other edge groups in Δ_i are either rigid vertex groups or edge groups in $\Gamma^i_{<d_0>}$.

(3) The other vertex groups in Δ_i that do not contain conjugates of $< p >$ nor $< q >$, and are not abelian star, are vertex groups in Γ_D^i, that are also vertex groups in $\Gamma^i_{<d_0>}$, which are not edge groups in Δ_i. The edge groups that are connected to these vertex groups are the edge groups that are connected to them in Γ_D^i (which are the same edge groups that are connected to these vertex groups in the graph of groups $\Gamma^i_{<d_0>}$).

Recall that the subgroup $< f, p, q >$ is the image of one of the limit groups, $Ipr_1, \ldots, Ipr_w$, and it does not factor through a free product in which $< p >$ is contained in one factor and $< q >$ is contained in another factor. Hence, if both subgroups $< p >$ and $< q >$ are non-trivial in $Tduo_i$, then $< f, p, q >$ intersects non-trivially some conjugates of the edge groups in the graph of groups Δ_i, i.e., those edge groups that correspond to edge groups or vertex groups in Γ_D^i, or abelian star edge groups. In this last case it intersects non-trivially conjugates of abelian star vertex groups in Δ_i. Let $V_i^1, \ldots, V_i^f$ be the conjugacy classes of intersections between the subgroup, $< f, p, q >$, and all the edge groups in Δ_i, and between $< f, p, q >$ and conjugates of the abelian star vertex groups in Δ_i.

Lemma 3.15. *Let $Tduo_i$ be one of the duo limit groups, $Tduo_1, \ldots, Tduo_m$, and suppose that both subgroups, $< p >$ and $< q >$, are non-trivial in $Tduo_i$. Then at least one of the subgroups, $V_i^1, \ldots, V_i^f$, intersects nontrivially either a conjugate of an abelian vertex group in Γ_D^i, or a conjugate of a rigid vertex group in $\Gamma^i_{<d_0>}$.*

Proof: Suppose that none of the subgroups V_j^i intersects non-trivially both the rigid vertex groups in $\Gamma^i_{<d_0>}$, and the abelian vertex groups that are connected to these rigid vertex groups in Γ_D^i.

We look at those subgroups, V_j^i which are the intersections of the subgroup, $< f, p, q >$, and conjugates of the abelian star vertex groups in Δ_i. These vertex groups must inherit non-trivial graphs of groups from Γ_D^i. Let V_j^i be one of these

vertex groups, and let Θ^i_j be its inherited graph of groups. The vertex groups in Θ^i_j are intersections of the subgroups V^i_j with an abelian vertex group in Γ^i_D or with a rigid vertex group in $\Gamma^i_{<d_0>}$. Since we assumed that such intersections are trivial, the vertex (and edge) groups in Θ^i_j are all trivial.

Hence, each of the subgroups, V^i_j, is a free group. Furthermore, recall that the subgroups V^i_j are conjugacy classes of intersections between the subgroup, $< f,p,q >$, and conjugates of subgroups that are obtained from rigid vertex groups in $\Gamma^i_{<d_0>}$ by amalgamation with abelian subgroups of the abelian subgroups $< e_1 >$ and $< e_2 >$. Therefore, in the graphs of groups that are inherited by the subgroups, V^i_j, from the graph of groups, Γ^i_D, we can separate those loops that are labeled by elements from the rigid vertex group and $< e_1 >$, or by elements of the rigid vertex group and $< e_2 >$. Those loops generate possibly trivial (free) factors in V^i_j that are contained in $< d_0, e_1 >$, and correspondingly in $< d_0, e_2 >$. Therefore, we obtain a (possibly trivial) free decomposition for each of the subgroups, V^i_j, $V^i_j = A_{<d_0,e_1>} * B_{<d_0,e_2>} * C$, where: $A_{<d_0,e_1>} << d_0, e_1 >$ and $B_{<d_0,e_2>} << d_0, e_2 >$.

Combining these free decompositions of the subgroups, V^i_j, with the graphs of groups that is inherited by $< f,p,q >$ from the graph of groups Δ_i of the duo limit group $Tduo_i$, we obtain a non-trivial free decomposition of the subgroups, $< f,p,q >$, in which the subgroup, $< p >$, is contained in one factor, and the subgroup, $< q >$, is contained in a second factor, a contradiction to our assumption that the test sequences of specializations of the subgroups, $< f,p,q >$, that are images of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, do not factor through free products in which $< p >$ is contained in one factor and $< q >$ is contained in a second factor. Therefore, the intersection of at least one of the subgroups, V^i_j, and either a conjugate of one of the rigid vertex groups in $\Gamma^i_{<d_0>}$, or one of the abelian vertex groups in Γ^i_D, is non-trivial.

$\square$

By construction, edge groups in Δ_i are either edge groups or rigid vertex groups in $\Gamma^i_{<d_0>}$, or they are abelian star edge groups (that are adjacent to abelian star vertex groups). If a subgroup V^i_j is contained in a conjugate of an edge group or a rigid vertex group in $\Gamma^i_{<d_0>}$, the statements of propositions 3.4 and 3.13 remain valid. Suppose that V^i_j is contained in a conjugate of an abelian star vertex group in Δ_i.

In that case, the subgroup V^i_j inherits a (possibly trivial) graph of groups decomposition from the graph of groups that is associated with the abelian star vertex group of Δ_i, into which it can be conjugated. We denote this graph of groups Θ^i_j.

With Θ^i_j we associate a finite collection of (conjugacy classes of) subgroups of $Tduo_i$. In the graph of groups Θ^i_j each edge group is either trivial or abelian. In Θ^i_j we color all the edges that either have non-trivial edge groups or that are connected to abelian vertex groups in Θ^i_j (i.e., to abelian vertex groups that intersect V^i_j non-trivially). If we erase from Θ^i_j all the edge groups that are not colored (note that these edges must have trivial edge groups), we are left with finitely many (possibly none) maximal connected subgraphs of Θ^i_j, that are associated with with V^i_j. By lemma 3.15 for at least one index j, $j = 1, \ldots, f$, there exists such non-trivial connected subgraph that is associated with V^i_j.

By construction, with each such connected (colored) subgraph of Θ^i_j, we naturally associate its fundamental group, that we denote, $C^i_j(t)$ (which is a subgroup

of V_j^i), where t is the index of the connected component. As V_j^i is f.p. and $C_j^i(t)$ is a factor in a free decomposition of V_j^i, $C_j^i(t)$ is f.p. as well.
In particular, $C_j^i(t)$ is finitely generated. If we fix a generating set for $C_j^i(t)$, then by the construction of the colored components of Θ_j^i, there exists a finite collection of elements in $Tduo_i$, that we associate with $C_j^i(t)$, so that:

(i) each element in the fixed generating set of $C_j^i(t)$ can be expressed as a fixed word in the (fixed) finite set of elements from $Tduo_i$.

(ii) the fixed set of elements from $Tduo_i$ consists of elements from the abelian vertex groups of the corresponding abelian star vertex group, elements from (conjugates of) the vertex group from $\Gamma^i_{<d_0>}$ that appear in the connected subgraph of Θ_j^i, and elements from (conjugates of) the vertex group from $\Gamma^i_{<d_0>}$ that appear in the connected subgraph, that are determined up to their left or right coset of one of the associated abelian groups, or up to a double coset of two (possibly the same) of the associated abelian groups.

Hence, if we fix a generating set for $C_j^i(t)$, we can associate with $C_j^i(t)$ finitely many elements from conjugates of the associated vertex group in $\Gamma^i_{<d_0>}$, finitely many left, right, and double cosets of such elements, and primitive roots of the images of the abelian groups that appear in such connected components.

Given each specialization of a subgroup, $C_j^i(t)$, that is associated with an equivalence class of $E(p,q)$, we look at the subgroup generated by this specialization together with primitive roots of the specializations of the abelian edge groups in the graph of groups that is associated with $C_j^i(t)$. Given such a subgroup (in the coefficient group F_k), we further look at its associated almost shortest specializations (with respect to the modular group of the connected subgraph of groups that is associated with $C_j^i(t)$). See definition 2.8 in [Se3], for almost shortest specializations. By our standard techniques (section 5 in [Se1]), The collection of such almost shortest specializations that are associated with $C_j^i(t)$ and all the equivalence classes of $E(p,q)$, factor through finitely many limit groups that we denote, $H_j^i(t)_r$.

Proposition 3.16. *Let $Tduo_i$ be one of the Duo limit groups, $Tduo_1, \dots, Tduo_m$, and let Δ_i be its associated graph of groups. Suppose that both subgroups, $< p >$ and $< q >$, are non-trivial in $Tduo_i$, and let the subgroup $< f,p,q >$, be the image in $Tduo_i$ of one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$. Since $< f,p,q >$ does not admit a free product in which the subgroups $< p >$ and $< q >$ can be conjugated into distinct factors, the subgroup $< f,p,q >$ intersects non-trivially some conjugates of the edge groups in the graph of groups Δ_i. We denoted the conjugacy classes of these intersection subgroups, and the conjugacy classes of intersections between the subgroup, $< f,p,q >$, and the abelian star vertex groups in Δ_i, $V_1^i, \dots, V_f^i$. By lemma 3.15 at least one of the subgroups, $V_1^i, \dots, V_f^i$, intersects nontrivially either a conjugate of an abelian vertex group in Γ_D^i, or a conjugate of a vertex group in $\Gamma^i_{<d_0>}$.*

The subgroups V_j^i inherit graph of groups decompositions from the abelian decompositions of the edge groups that they intersect. With these graphs of groups we associated certain connected subgraphs, and we denoted their fundamental groups, $C_j^i(t)$ (where t is the index of the connected subgraph). Given each specialization of a subgroup, $C_j^i(t)$, we add primitive roots of the specializations of the abelian edge groups in the graph of groups decomposition that is associated with $C_j^i(t)$, and

looked at the associated almost shortest specializations (with respect to the modular group of the connected subgraph), and the collection of almost shortest specializations factor through finitely many limit groups that we denoted, $H^i_j(t)_r$.

Let $G_{j(i)}$ be the graded completion from the Diophantine envelope of $E(p,q)$, $G_1,\ldots,G_t$, that is associated with $Tduo_i$, i.e., the graded completion from which $Tduo_i$ was constructed. Then there exists a global integer $b_i > 0$, so that for any rigid or a family of strictly solid specializations of the terminal rigid or solid limit group of the graded completion, $G_{j(i)}$, there are at most b_i rigid or strictly solid families of specializations of the terminal rigid or solid limit group of $Tduo_i$ that extend the given family of specializations of the terminal limit group of $G_{j(i)}$, and so that these rigid or strictly solid families can be extended to generic rigid and strictly solid specializations of $< f,p,q >$. Furthermore, these specializations of the terminal limit group of $Tduo_i$ restrict to at most b_i conjugacy classes of almost shortest specializations that factor through the subgroups, $H^j_i(t)_r$. Equivalently, these specializations of the terminal limit group of $Tduo_i$ restrict to at most b_i conjugacy classes of specializations of finitely many (fixed) elements, left, right and double cosets, of specializations of another finite set of elements, that are associated with each of the subgroups, $C^i_j(t)$, that are associated with the connected components of the graphs of groups, Θ^i_j.

Proof: The bound on the number of conjugacy classes of specializations of rigid vertex groups in the graph of groups, $\Gamma_{<d_0>}$, follows from the existence of a uniform bound on the number of strictly solid families of specializations of a solid limit group with a fixed value of the defining parameters (i.e., a bound that does not depend on the specific value of the defining parameters) that was proved in theorem 2.9 in [Se3]. Given this uniform bound, the bound on the number of almost shortest specializations that factor through the limit groups, $H^i_j(t)_r$, follows by construction. This last bound is equivalent to a bound on the number of conjugacy classes of specializations of finite collection of elements, and a bound on the number of possible right, left and double cosets of another collection of finitely many elements, that together generate the subgroups, $C^i_j(t)$.

$\square$

Suppose that there exist a duo limit group $Tduo$, and an equivalence class of $E(p,q)$, for which there exists an infinite sequence of conjugacy classes of almost shortest specializations (to which we have added primitive roots of specializations of the abelian edge groups in the graphs of groups that are associated with the subgroups $C^i_j(t)$), that factor through the various limit groups, $H^i_j(t)_r$, that can be extended to couples of test sequences of the two graded completions that are associated with $Tduo$, so that restrictions of generic elements in these test sequences, (f_n,p_n,q_n), prove that the couples $(p_n,q_n) \in E(p,q)$, these test sequences restrict to valid proofs that the couples $(p_n,q_0(n))$ and $(q_0(n),q_n)$ belong to $E(p,q)$ (recall that $q_0(n)$ is the restriction of the specializations $d_0(n)$ to the elements q_0), and furthermore these test sequences restrict to sequences of distinct couples, $\{(p_n,q_n)\}$.

In that case we can associate with the entire collection of such equivalence classes of $E(p,q)$, a finite collection of duo limit groups, $Sduo_1,\ldots,Sduo_h$. These limit groups are constructed in a similar way to the way they were constructed in case the graded completions that are associated with the duo limit groups, $Tduo_1,\ldots,Tduo_m$, contain no abelian vertex groups in any of their levels, just that in the construction we add to the specializations of the subgroups, $V^i_1,\ldots,V^i_f$,

specializations of the subgroups, $H^i_j(t)_r$, with elements that demonstrates that the specializations of certain subgroups of $H^i_j(t)_r$ are conjugate or in the same coset or double coset of specializations of corresponding subgroups in $C^i_j(t)$ (recall that the specializations of $H^i_j(t)_r$ are almost shortest specializations, that are obtained from specializations of the subgroups $C^i_j(t)$ by using the modular groups that are associated with the subgroups $C^i_j(t)$, together with primitive roots of specializations of abelian edge groups in the graphs of groups that are associated with the subgroups, $C^i_j(t)$).

Given the finite collection of duo limit groups, $Sduo_1, \ldots, Sduo_h$, we can associate with them finitely many uniformization limit groups, precisely as we did in case the graded completions that are associated with the duo limit groups, $Sduo_1, \ldots, Sduo_h$, contain no abelian vertex groups in any of their levels. Note that the graded completions that are associated with the duo limit groups, $Sduo_1, \ldots, Sduo_h$, are replaced by some closures in their associated uniformization limit groups.

With the terminal limit group of a uniformization limit group, $Unif$, we can associate graphs of groups (with abelian edge groups), Γ_D and $\Gamma_{<d_0>}$, precisely as we did with the terminal limit groups of the duo limit groups, $Tduo$. Furthermore, from the graph of groups Γ_D we can construct a graph of groups Δ of the ambient uniformization limit group, $Unif$. The edge group in Δ include all the groups that are obtained from rigid vertex groups in $\Gamma_{<d_0>}$ that are amalgamated with abelian vertex groups that are connected to them in Γ_D. The image, $<\tilde{f}, p, \tilde{q}>$, of the map from one of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, into the uniformization limit group, $Unif$, is assumed not to factor through a free product in which $<p>$ is contained in one factor and $<q>$ can be conjugated into a second factor. Hence, if both subgroups $<p>$ and $<\tilde{q}>$ are non-trivial in $Unif$, then the subgroup, $<\tilde{f}, p, \tilde{q}>$, intersects non-trivially some conjugates of the edge groups in its associated graph of groups, Δ.

Let $\tilde{V}^i_j$ be the conjugacy classes of the intersections between the subgroup $<\tilde{f}, p, \tilde{q}>$ and the edge groups in Δ_i, the graph of groups that is associated with the uniformization limit group $Unif_i$. Clearly, lemma 3.15 remains valid for the subgroups, $\tilde{V}^i_j$, hence, at least one of these subgroups intersects non-trivially a conjugate of a rigid vertex group in $\Gamma_{<d_0>}$, or an abelian vertex group that is connected to one of the rigid vertex groups in $\Gamma_{<d_0>}$. We can further associate a graph of groups decomposition with each of the subgroups $\tilde{V}^i_j$, that is inherited from the graph of groups, Γ_D. In these graphs of groups we look at connected components, precisely as in the construction of the groups $C^i_j(t)$, and we set $\tilde{C}^i_j(t)$ to be the fundamental groups of these connected components. With each specialization of the groups, $\tilde{C}^i_j(t)$, we associate its finite collection of almost shortest specializations (under the action of the modular groups of the connected subgraphs), to which we add primitive roots of specializations of abelian edge groups in the graphs of groups that are associated with the subgroups, $\tilde{C}^i_j(t)$. With the entire collection of almost shortest specializations (together with specializations of their associated primitive roots), we associate finitely many limit groups, that we denote, $\tilde{H}^i_j(t)_r$.

Suppose that there exists a uniformization limit group $Unif$, and an equivalence class of $E(p,q)$, for which there exists an infinite sequence of conjugacy classes of almost shortest specializations (to which we have added primitive roots of specializations of abelian edge groups), that factor through the various limit groups,

$\tilde{H}^i_j(t)_r$, that can be extended to couples of test sequences of the two graded completions that are associated with $Unif$, so that restrictions of generic elements in these test sequences, $(\tilde{f}_n, p_n, \tilde{q}_n)$, prove that the couples $(p_n, \tilde{q}_n) \in E(p,q)$, these test sequences restrict to valid proofs that the couples $(p_n, q_0(n))$ and $(q_0(n), \tilde{q}_0(n))$ belong to $E(p,q)$ (recall that $q_0(n)$ is the restriction of the specializations $d_0(n)$ to the elements q_0), and furthermore these test sequences restrict to sequences of distinct couples, $\{(p_n, \tilde{q}_n)\}$.

In that case we can associate with the entire collection of such equivalence classes of $E(p,q)$, a finite collection of duo limit groups, $Sduo_1, \ldots, Sduo_h$, and with each of these duo limit groups we can associate a finite collection of uniformization limit groups, precisely as we did in the previous steps, and in a similar way to the construction in case the graded completions that are associated with the uniformization and duo limit groups contain no abelian vertex groups in any of their levels.

We continue our constructions iteratively. In each step we construct new uniformization limit groups, $Unif^i$. If the image of both subgroups, $< p >$ and $< q >$, in $Unif^i$, are non-trivial, then the image of the map from one of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, into $Unif^i$ (that we denoted $< f_i, p, q_i >$), intersects non-trivially some conjugates of the edge groups in the graph of groups Δ^i that is associated with $Unif^i$. We can further associate graphs of groups with each such intersection subgroup, and look at connected components in these graphs of groups as we did in previous steps of the procedure.

If there are equivalence classes of $E(p,q)$ for which there are infinitely many conjugacy classes of almost shortest specializations that are associated with connected components in the constructed graphs of groups, to which we add specializations of primitive roots of the associated abelian edge groups, we continue the next step. The procedure terminates by an argument which is similar to the one that was used in proving theorem 3.9.

Theorem 3.17. *The iterative procedure for the construction of the uniformization limit groups, $Unif^i$, in the general case (i.e., when the graded completions that are associated with the uniformization limit groups may contain abelian vertex groups along their levels), terminates after finitely many steps.*

Proof: Suppose that the procedure for the construction of uniformization limit groups does not terminate. Since at each step we construct only finitely many uniformization limit groups, there must exist an infinite path of uniformization limit groups that is constructed along the iterative process.

First, suppose that along such an infinite path, there exists an infinite sequence of primitive roots of (colored) abelian edge and vertex groups in the graphs of groups, $\Theta^i_j(t)$, that are non-elliptic in the next levels, i.e., these primitive roots have infinitely many specializations that are associated with a given class. In this case we get a contradiction to theorem 1.3 in [Se3], in a similar way to the contradiction that was obtained in proving theorem 3.9. Since at each level there are only finitely many edges with non-trivial abelian edge groups (in the graphs of groups, $\Theta^i_j(t)$, that are associated with the uniformization limit group that is constructed in this step along the infinite path), we get a sequence of homomorphisms from the subgroups $< f_i, p, q_i >$, that are associated with the uniformization limit groups that are constructed along the path, into the coefficient group F_k, that converges into an action of some limit group L on some real tree. Since there are infinitely many

rates of growth for these primitive roots, the sequence of homomorphisms that we look at contains no infinite subsequence that converges into a free actions on some R^m-tree, for any positive integer m. This contradicts theorem 1.3 in [Se3].

Hence, after finitely many steps along the infinite path, the primitive roots of the abelian edge and vertex groups in the graphs of groups, $\Theta^i_j(t)$, remain elliptic throughout the infinite path.

Suppose that there exists an infinite subsequence along the infinite path, for which to the completions in the duo limit group $Sduo_i$ from which the uniformization limit groups, $Unif^i$, are constructed, either non-abelian vertex groups are being added, or abelian vertex groups are being added, and these abelian vertex groups are not part of the associated graphs of groups, Γ^i_D, i.e., they are not part of the vertex groups $< d^i_0, e^i_1, e^i_2 >$ that are associated with the uniformization limit groups, $Unif^i$, for the indices i in the subsequence. Suppose further that the subgroups $< f_i, p, q_i >$ that are associated with the uniformization limit groups, $Unif^i$, or rather some of their subgroups, inherit non-trivial abelian decompositions from these additional non-abelian and abelian vertex groups. Then, once again, we are able to construct a sequence of homomorphisms from the subgroups, $< f_i, p, q_i >$, into the coefficient group, F_k, that converges into an action of a limit group, L, on some real tree, and the elements in this limit group L, have infinitely many rates of growth, in contradiction to theorem 1.3 in [Se3], that guarantees the existence of a subsequence that converges into a free action of L on some R^m-tree for some integer m.

Hence, after finitely many steps along the infinite path, we may assume that new abelian vertex groups are being added to the graphs of groups, Γ^i_D, and along a subsequence, the subgroups, $< f_i, p, q_i >$, or rather their associated subgroups, $C^j_i(t)$, inherit new abelian decompositions from these additional abelian vertex groups. We may also assume that all the abelian edge groups in the graphs of groups, $C^i_j(t)$, remain elliptic in the graphs of groups that are associated with the components, $C^{i'}_j(t)$, for $i' > i$.

Therefore, the new vertex and edge groups that are being added to the graphs of groups that are associated with the components, $C^i_j(t)$, along the infinite path, further and further refine these graphs of groups, which contradicts theorem 1.3 in [Se3], or alternatively contradicts the accessibility for small splittings of finitely presented groups of M. Bestvina and M. Feighn [Be-Fe1]. Therefore, every path along the iterative procedure is finite, so the iterative procedure has to terminate after finitely many steps.

□

After the procedure terminates we are left with finitely many uniformization limit groups that were constructed in its various steps. For each equivalence class which is not one of the finitely many that were excluded in theorem 3.1, there exists some uniformization limit group, $Unif_i$, so that the almost shortest specializations that are associated with the equivalence class and with the connected components, $C^i_j(t)$, that are associated with the uniformization limit group, $Unif_i$, belong to boundedly many conjugacy classes. Hence, if we fix generating sets for the subgroups that are associated with these connected components, then up to conjugacy the elements in these generating sets can be written as words in elements that either belong to a bounded set (where the bounded set depends on the equivalence class), or belong to a bounded set of right, left or double cosets of cyclic groups (where the cosets

and the cyclic groups depend on the equivalence class).

To obtain from this final conclusion a *separation of variables* in the general case (i.e., in the presence of abelian vertex groups in the graded completions that are associated with duo and uniformization limit groups), we still need to generalize lemmas 3.12 and 3.14.

With each uniformization limit group, $Unif_i$, that is obtained as a terminal uniformization limit group of the iterative process for the construction of uniformization limit groups, as it is in particular a duo limit group, there are associated graphs of groups, $\Gamma^i_{<d_0>}$, Γ^i_D, and Δ_i, precisely as the corresponding graphs of groups that are associated with the duo limit groups, $Tduo_1, \ldots, Tduo_m$. Recall that the edge groups in the graph of groups, Δ_i, are either rigid vertex groups or edge groups in $\Gamma^i_{<d_0>}$, or they are abelian star edge groups, i.e, they are the fundamental group of a subgraph of groups in Γ^i_D, that contain one rigid vertex group from $\Gamma^i_{<d_0>}$ and several abelian vertex that are connected to this vertex group (which are subgroups of $< e_1 >$ or $< e_2 >$).

Let Λ'_i be the graph of groups decomposition that the subgroup, $< f_i, p, q_i >$, inherits from the graph of groups, Δ_i. Let V^i_j be the (conjugacy classes of) intersections between the subgroup, $< f_i, p, q_i >$, and conjugates of the abelian star vertex groups in Δ_i, i.e., vertex groups that are the fundamental group of a graph of groups that contains a rigid vertex group from the abelian decomposition $\Gamma^i_{<d_0>}$, and abelian vertex groups that are connected to it in Γ^i_D. Each of the subgroups, V^i_j, inherits a graph of groups decomposition from the graph of groups that is associated with the corresponding abelian star vertex group in Δ_i. We denote the graph of groups that is inherited by the subgroups, V^i_j, from the graphs of groups of the associated abelian star vertex groups in Δ_i, Θ^i_j.

Recall that in the graphs of groups, Θ^i_j, we colored edges for which the subgroup V^i_j contains a non-trivial element that commutes with the associated edge group. This adds notation to some of the edges in Θ^i_j. In Θ^i_j we look at maximal connected subgraphs in which all the edges are colored. This associates finitely many (possibly none) subgraphs with Θ^i_j, and we denoted the fundamental groups of these connected subgraphs, $C^i_j(t)$ (t is the index of the connected subgraph). We set the graph of groups ${\Theta'}^i_j$ to be the graph of groups that is obtained from Θ^i_j by collapsing the subgraphs of groups that are associated with the subgroups $C^i_j(t)$. Note that all the edge groups in ${\Theta'}^i_j$ are trivial, and all the non-trivial vertex groups are (conjugates of) the subgroups, $C^i_j(t)$.

From the graphs of groups Θ^i_j and ${\Theta'}^i_j$, and the graph of groups Λ'_i, we construct a graph of groups that we denote, Λ_i. The graphs of groups Θ^i_j and ${\Theta'}^i_j$ provide graphs of groups for the subgroups, V^i_j, which are intersections of the subgroups, $< f_i, p, q_i >$, with abelian star edge groups in the graph of groups, Δ_i.

In the graphs of groups Θ^i_j and ${\Theta'}^i_j$, it is possible to identify the edge groups that are connected to the vertex group V^i_j in the graph of groups, Λ'_i. Indeed these edge groups are contained in the fundamental groups of subgraphs of groups in ${\Theta'}^i_j$, where these subgraphs may intersect only in the vertex groups that are associated with the factors, $C^i_j(t)$. Hence, we may use the graphs of groups, ${\Theta'}^i_j$, and the graph of groups, Λ'_i, to obtain a refinement of the graph of groups, Λ'_i, that we denote, Λ_i.

To obtain the graph of groups, Λ_i, we first modify the graphs of groups, ${\Theta'}^i_j$. We replace ${\Theta'}^i_j$, by a graph of groups, α^i_j, as follows. The vertex groups in the graphs of groups, α^i_j, are (maximal) subgraphs of groups in ${\Theta'}^i_j$ for which their fundamental groups can be generated by edge groups in Λ'_i, together with the vertex groups, $C^i_j(t)$, that lie on these subgraphs. The graph of groups, α^i_j, is obtained from ${\Theta'}^i_j$ by collapsing these subgraphs into vertices. Edge groups in α^i_j are either the subgroups, $C^i_j(t)$, which are vertex groups in ${\Theta'}^i_j$, or they are trivial. Since all the edge groups in ${\Lambda'}_i$ can be conjugated into vertex groups in the graphs of groups α^i_j, the graphs of groups, ${\Lambda'}_i$ and α^i_j, have a common refinement, that we denote Λ_i. Finally, since the subgroup, $< f_i, p, q_i >$, does not factor through a non-trivial free product in which the subgroups, $< p >$ and $< q >$ are contained in factors, none of the edge groups in the graphs of groups, α^i_j, are trivial.

Lemma 3.18. *Let Λ_i be the graph of groups decomposition that is the common refinement of the graphs of groups, ${\Lambda'}_i$ and α^i_j, that are inherited by the subgroup, $< f_i, p, q_i >$, and its subgroups V^i_j, from the uniformization limit group, $Unif_i$. Then Λ_i is either:*

1. *Λ_i is a trivial graph, i.e., a graph of groups that contains a single vertex. In that case either the subgroup $< p >$ or the subgroup $< q_i >$ is contained in the distinguished vertex group in $\Gamma^i_{<d_0>}$. Hence, for each equivalence class there are at most boundedly many values of specializations of either the subgroup $< p >$ or the subgroup $< q_i >$ (where the bound on the number of specializations that is associated with each equivalence class is uniform).*
2. *Λ_i has more than one vertex group, and the subgroups $< p >$ and $< q_i >$ are contained in the same vertex group. In that case either $< p >$ or $< q_i >$ are contained in the distinguished vertex group in $\Gamma^i_{<d_0>}$, hence, the conclusion of part (i) holds, and there are at most boundedly many specializations of either $< p >$ or $< q_i >$ that are associated with each equivalence class.*
3. *Λ_i has more than one vertex group, and either $< p >$ or $< q_i >$ are contained in a vertex group in Λ_i, which is stabilized by one of the subgroups, $C^i_j(t)$, which is a subgraph of one of the graph of groups, Θ^i_j. By proposition 3.16, the subgroup $C^i_j(t)$ that contains $< q_i >$ (or $< p >$) is generated by elements that are (fixed) words in elements that belong to boundedly many left, right, and double cosets of boundedly many cyclic groups (the cosets and the cyclic groups depend only on the equivalence class, and the bound on their possible number is uniform).*
4. *Λ_i has more than one vertex, $< p >$ is contained in the stabilizer of one vertex and $< q_i >$ is contained in the stabilizer of another vertex in Λ_i. The edge group in Λ_i that connects between the vertex that is stabilized by $< p >$ and the vertex that is stabilized by $< q_i >$ may be trivial. The other edge groups in Λ_i (that are all non-trivial), are either conjugates of subgroups of edge or vertex groups in the graph of groups, $\Gamma^i_{<d_0>}$ (the graph of groups for the terminal rigid or solid limit group on the uniformization limit groups, $Unif_i$), or they are edge groups in one of the graphs of groups, α^i_j, which are all conjugates of the subgroups, $C^i_j(t)$, or they are edge groups in the graph of groups Δ_i that can be conjugated into vertex groups in the graphs of groups, α^i_j. In this last case (an edge group in Δ_i that can be conjugated*

into a vertex group in α^i_j), the vertex group into which the edge group in Δ_i is connected, admits a (possibly trivial) free product: $M_1 * \ldots * M_d * F$. In this free product F is a free group, and all the factors $M_1, \ldots, M_s$ are conjugates of some of the subgroups, $C^i_j(t)$. By proposition 3.16, each of the factors M_e is generated by finitely many elements, and each of these elements can be written as a fixed word in elements that belong to boundedly many left, right, and double cosets of boundedly many cyclic groups (the cosets and the cyclic groups depend only on the equivalence class, and the bound on their possible number is uniform).

Proof: Suppose that the subgroups $< p >$ and $< q_i >$ fix the same vertex in Λ_i. By the structure of the graph of groups, Λ_i, either $< p >$ or $< q_i >$ is contained in an edge group that is connected to this vertex group, which implies that either $< p >$ or $< q_i >$ must be contained in the distinguished vertex group in $\Lambda^i_{<d_0>}$. Hence, either $< p >$ or $< q_i >$ admits only boundedly many values that are associated with each equivalence class, and the bound on the number of values that is associated with each equivalence class is uniform (it doesn't depend on the class). This proves parts (1) and (2). Part (3) follows from proposition 3.16.

Since the subgroup, $< f_i, p, q_i >$, admits no free decomposition in which the subgroups, $< p >$ and $< q_i >$, are contained in the same factor, all the edge groups of Λ_i, except perhaps the edge group that connects between the vertex that is stabilized by $< p >$ to the vertex that is stabilized by $< q_i >$, must be non-trivial. By the construction of the graph of groups Λ_i, an edge group can be either conjugated into an edge group or a vertex group in $\Gamma^i_{<d_0>}$, or it can be conjugated into one of the subgroups, $C^i_j(t)$, or it can be conjugated into a vertex group in one of the graphs of groups, α^i_j. The graphs of groups α^i_j are obtained from the graphs of groups, ${\Theta'}^i_j$, and each vertex group in α^i_j is obtained as a (possibly trivial) free product of some conjugates of the subgroups, $C^i_j(t)$. Hence, each vertex group in Λ_i that is not stabilized by $< p >$ nor by $< q >$, and can not be conjugated into a vertex group or an edge group in $\Gamma^i_{<d_0>}$, must be a vertex group in one of the graphs of groups, α^i_j, and these, by construction, admit a (possibly trivial) free decomposition of the form that is presented in part (4) of the lemma.

□

The graph of groups, Λ_i, lemma 3.18, and the uniform bounds on the number of conjugacy classes and left, right and double cosets of cyclic groups of elements that determine sets of generators for the subgroups, $C^i_j(t)$, can be viewed once again as a separation of variables in the general case (i.e., in case the terminal limit groups of all the duo and uniformization limit groups that are constructed through our iterative procedure may be rigid or solid, and the graded completions that are associated with these groups may contain abelian vertex groups in any of their levels). In the next section we use this separation of variables to associate parameters with the equivalence classes of a definable equivalence relation, $E(p, q)$.

§4. Equivalence Relations and their Parameters

In the first section of this paper we have constructed the Diophantine envelope of a definable set (theorem 1.3), and then used it to construct the Duo envelope of a definable set (theorem 1.4).

Recall that by its definition (see definition 1.1), a Duo limit group Duo admits an amalgamated product: $Duo = < d_1, p > *_{<d_0,e_1>} < d_0, e_1, e_2 > *_{<d_0,e_2>} < d_2, q >$ where $< e_1 >$ and $< e_2 >$ are free abelian groups with pegs in $< d_0 >$, i.e., free abelian groups that commute with non-trivial elements in $< d_0 >$. A specialization of the parameters $< d_0 >$ of a Duo limit group gives us a Duo family of it.

To analyze definable equivalence relations over a free (or a hyperbolic) group, our strategy is to further study the parameters $(< d_0 >)$ that are associated with the Duo families that are associated with the Duo limit groups that form the Duo envelope of a definable equivalence relation.
In the previous section we modified and analyzed the construction of the Duo envelopes that were presented in theorem 1.4, in the special case of a definable equivalence relation. We further carefully studied the set of values of the parameters that are associated with the duo families that are associated with each equivalence class. This careful study, that uses what we called $uniformization$ limit groups that we associated with the Duo envelope, enabled one to associate a "bounded" set of (values of) certain subgroups of the parameters that are associated with the Duo families of the Duo envelope, for each equivalence class of a definable equivalence relation (the bounded set of values of the subgroups of parameters is modulo the basic imaginaries that were presented in the second section).

The bounds that we achieved on the number of conjugacy classes, left, right, and double cosets of cyclic groups that are associated with each equivalence class, allowed us to obtain what we view as "separation of variables". This means that with the original subgroups of parameters, $< p >$ and $< q >$, we associate a bigger subgroup, for which there exists a graph of groups decomposition, where $< p >$ is contained in one vertex group, $< q >$ is contained in a second vertex group, and either the number of specializations of edge groups (up to the imaginaries that were presented in section 2) is bounded for each equivalence class of $E(p,q)$ (see lemmas 3.12,3.14 and 3.18), where the bound does not depend on the specific equivalence class, or as in part (4) of lemma 3.18, an edge group is contained in a vertex group, and this vertex group is a free product of subgroups that are either free, or generated by elements with boundedly many possible specializations (up to the basic imaginaries) that are associated with each equivalence class.

However, the parameters that we associated with each equivalence class (that are bounded up to the imaginaries that were presented in section 2), do not separate between equivalence classes in general. To obtain subgroups of parameters that have the same types of bounds as the ones constructed in the previous section, that do separate between classes, we present a new iterative procedure that uses both the sieve procedure [Se6] (that was used for quantifier elimination) together with the procedure for separation of variables that was presented in the previous section. The combined procedure is a (new) sieve procedure that preserves the separation of variables along its various steps, and its termination (that follows from the termination of the sieve procedure and the procedure for the separation of variables), produces the desired subgroups of parameters, that do separate between classes and admit boundedly many values (up to the imaginaries of section 2) for each equivalence class (where the bound on the number of values does not depend on the specific equivalence class).

Let $F_k = < a_1, \dots, a_k >$ be a non-abelian free group, and let $E(p,q)$ be a definable equivalence relation over F_k. With the definable equivalence relation, $E(p,q)$,

being a definable set, one associates using theorems 1.3 and 1.4, a Diophantine and a Duo envelopes. Let $G_1, \ldots, G_t$ be the Diophantine envelope of the given definable equivalence relation $E(p,q)$, and let $Duo_1, \ldots, Duo_r$, be its Duo envelope.

Recall that with the definable equivalence relation, $E(p,q)$, being a definable set, one associates (using the sieve procedure for quantifier elimination [Se6]) finitely many (terminal) rigid and solid limit groups, $Term_1, \ldots, Term_s$. With each of the terminal limit groups $Term_i$ there are 4 sets associated, $B_j(Term_i)$, $j = 1, \ldots, 4$, and the definable set $E(p,q)$ is the set:

$$E(p,q) \; = \cup_{i=1}^{s} \left(B_1(Term_i) \setminus B_2(Term_i)\right) \cup \left(B_3(Term_i) \setminus B_4(Term_i)\right).$$

Given this finite set of terminal limit groups, $Term_1, \ldots, Term_s$, it is possible to demonstrate that a couple, $(p,q) \in E(p,q)$, using a specialization of one out of finitely many limit groups, that we denoted: $< x, y, u, v, r, p, q, a >$, where each of these limit groups is generated by the subgroup $< p, q >$, together with elements x for rigid and strictly solid specializations of some of the terminal limit groups, $Term_1, \ldots, Term_s$, elements y, u, v for rigid and strictly solid specializations of some of the terminal limit groups of the Non-Rigid, Non-Solid, Left, Root, Extra PS, and Generic Collapse Extra PS resolutions that are associated with some of these terminal limit groups, and elements for specializations of primitive roots of the specializations of edge groups in the graded abelian decomposition of some of the terminal limit groups, $Term_1, \ldots, Term_s$, and in the graded abelian decompositions of the terminal limit groups of some of the Extra PS resolutions that are associated with them (see the proof of theorem 1.3).

Theorem 3.1 associates with the given definable equivalence relation, $E(p,q)$, finitely many rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, so that apart from finitely many equivalence classes, for each couple, $(p,q) \in E(p,q)$, there exists a rigid or a strictly solid family of homomorphisms from at least one of the limit groups, $Ipr_1, \ldots, Ipr_w$, to the coefficient group F_k, so that the rigid homomorphisms or the strictly solid homomorphisms from the given strictly solid family do not factor through a free product $A * B$ in which $< p >< A$ and $< q >< B$, and each of these homomorphisms restricts to a valid proof that $(p,q) \in E(p,q)$, i.e., restricts to a specialization of one of the limit groups, $< x, y, u, v, r, p, q, a >$, that demonstrates that $(p,q) \in E(p,q)$.

To obtain separation of variables in the previous section, we started with the Diophantine envelope of the given definable equivalence relation, $G_1, \ldots, G_t$. With each graded completion G_j, $1 \leq j \leq t$, we associated a finite collection of duo limit groups. First, we collected all the test sequences of the completion G_j, that can be extended to rigid or strictly solid specializations of one of the rigid or solid limit groups, $Ipr_1, \ldots, Ipr_w$, that restrict to valid proofs that the corresponding couples, $\{(p_n, q_n)\}$, are in the equivalence relation $E(p,q)$. We further required that these test sequences of specializations can not be factored through a free product in which $< p >$ is contained in one factor, and $< q >$ is contained in the second factor. This collection of test sequences of the graded completions, $G_1, \ldots, G_t$ that can be extended to specializations of $Ipr_1, \ldots, Ipr_w$, can be collected in finitely many duo limit groups (using the techniques that were used for collecting formal solutions in [Se2]). Then we used the sieve procedure [Se6] to construct finitely many duo limit groups, $Tduo_1, \ldots, Tduo_m$, that still collect all these extended test sequences of the graded completions, $G_1, \ldots, G_t$, and for which there exist generic points (i.e., duo

test sequences) that restrict to valid proofs that the restricted couples, $\{(p_n, q_n)\}$, are in the given equivalence relation $E(p,q)$.

The collection of duo limit groups, $Tduo_1, \dots, Tduo_m$, is the starting point for the iterative procedure for separation of variables. With them we associated iteratively a collection of uniformization limit groups, $Unif_1, \dots, Unif_v$, until the iterative procedure terminates, and for each equivalence class of $E(p,q)$, apart from the finitely many equivalence classes that are singled out in theorem 3.1, there exists at least one uniformization limit group, $Unif_i$, that satisfies the conclusions of lemma 3.18. Recall that by lemma 3.18, the image of the subgroup, $Ipr_1, \dots, Ipr_w$ that is associated with the uniformization limit group, $Unif_i$, that we denoted, $< f_i, p, q_i >$, inherits a graph of groups decomposition Λ_i from the graph of groups decomposition Δ_i of the ambient uniformization limit group, $Unif_i$. For each equivalence class of $E(p,q)$ that is associated with $Unif_i$, an edge group in Λ_i is either generated by finitely many elements that admit boundedly many values (that are associated with the equivalence class) up to conjugation, and right, left and double cosets of boundedly many cyclic groups (see parts (1)-(3) in lemma 3.18), or an edge group is embedded into a vertex group in Λ_i, and this vertex group in Λ_i is a free product of a free group and finitely many factors, so that each of the factors is generated by finitely many elements that admit boundedly many values (that are associated with the equivalence class) up to conjugation, and right, left and double cosets of boundedly many cyclic groups (see part (4) in lemma 3.18). Furthermore, the subgroups $< p >$ and $< q >$ are both contained in vertex groups in Λ_i (lemma 3.18).

The edge groups in the graphs of groups, Λ_i, enable one to associate parameters with each equivalence class of $E(p,q)$, where these parameters admit only boundedly many families of values for each equivalence class, where each family is defined using the imaginaries that are presented in section 2 (i.e., each family is defined up to conjugation and right, left and double cosets of cyclic groups), The parameters that come from the edge groups in the graphs of groups Λ_i that are associated with each equivalence class, and are bounded up to the imaginaries that are presented in section 2 (i.e., up to conjugation and right, left and double cosets of cyclic groups), are not guaranteed to separate between equivalence classes in general. To use the graphs of groups Λ_i to obtain parameters that do separate between equivalence classes, we use the graphs of groups Λ_i as a first step in an iterative procedure that combines the procedure for separation of variables (that was presented in the previous section), with the sieve procedure for quantifier elimination that was presented in [Se6].

For presentation purposes, as we did in the previous section, we start by presenting the combined procedure assuming that the graded closures that are associated with all the duo limit groups that were used in the construction of the uniformization limit groups, $Unif_i$, and the graphs of groups Λ_i, do not contain abelian vertex groups in any of their levels. Later on we modify the procedure to omit this assumption.

We continue with the uniformization limit groups, $Unif_i$, in parallel. Hence, for brevity, we denote the uniformization limit group that we continue with, $Unif$. By construction, with each such uniformization limit group, $Unif$, there is an associated subgroup, $< f, p, q >$, (which is the image of one of the rigid and solid limit groups, $Ipr_1, \dots, Ipr_w$), and graph of groups decomposition, Λ. Recall that

under the assumption that there are no abelian vertex groups in any of the levels of the graded closures that are associated with the uniformization limit group, $Unif$, by lemma 3.14 the edge group in Λ that connects the vertex that is stabilized by $<p>$ with the vertex that is stabilized by $<q>$ may be trivial (only in case there are more edges that connect these two vertices), and that for each equivalence class of $E(p,q)$ that is associated with $Unif$, the associated values of the the other edge groups in Λ belong to boundedly many conjugacy classes (lemma 3.14).

If both subgroups, $<p>$ and $<q>$, are contained in the same vertex group in Λ, then for each equivalence class that is associated with $Unif$, there are only boundedly many associated values of either the subgroup $<p>$ or the subgroup $<q>$, and these values that belong to the equivalence class obviously determine the class. Hence, we can assume that the graph of groups Λ_i contains at least two vertex groups, and that $<p>$ is contained in one vertex group, and $<q>$ is contained in another vertex group in Λ.

In section 12 of [Se1] we presented the multi-graded Makanin-Razborov diagram. Recall that this multi-graded diagram encodes all the homomorphisms of a given limit group into a free group, if the specialization of a certain subgroup of the limit group is fixed, and the specializations of finitely many other subgroups is fixed up to conjugacy.
Let $<u,p>$ be the vertex group that contains $<p>$ in Λ, and let $<v,q>$ be the subgroup that contains $<q>$ in Λ. With both $<u,p>$ and $<v,q>$ we associate their taut multi-graded Makanin-Razborov diagram with respect to the edge groups in Λ_i that are connected to them (see section 12 in [Se1] for the construction of the multi-graded Makanin-Razborov diagram). We continue with all the possible pairs of taut multi-graded resolutions in these taut multi-graded Makanin-Razborov diagrams, $MGRes_u$ of $<u,p>$, and $MGRes_v$ of $<v,q>$, in parallel.

Given a pair of multi-graded resolutions, $MGRes_u$ of $<u,p>$ and $MGRes_v$ of $<v,q>$, we use the graph of groups, Λ, that is inherited from the splitting of the uniformization limit group, $Unif$, by its subgroup, $<f,p,q>$, and with the collections of specializations that factor through $MGRes_u$ and $MGRes_v$, we associate finitely many duo resolutions for the subgroup $<f,p,q>$. We denote each of the constructed duo resolutions, *DuoRes*. With such a duo resolution, *DuoRes*, we can associate the standard auxiliary resolutions that play a role in each step of the sieve procedure, i.e., Non-rigid, Non-solid, Left, Root, Extra, and Generic Collapse Extra resolutions (see sections 1 and 3 of [Se5] for the construction of these resolutions).
Before we continue with the analysis of the constructed duo resolutions, and the specializations that factor through them, we need the following simple lemma, which is similar to lemma 3.2.

Lemma 4.1. *Let DuoRes be a duo resolution that is constructed from two multi-graded resolutions, $MGRes_u$ and $MGRes_v$. Recall that by lemma 3.14, with each equivalence class of the definable equivalence relation , $E(p,q)$, we have associated a bounded collection of (conjugacy classes of) specializations of the edge groups in the graphs of groups, Λ, that are associated with the uniformization limit groups, $Unif$.*

Then for each possible specialization of the edge groups in the graph of groups, Λ, that is associated with the uniformization limit group, $Unif$, (that are defined up to

conjugacy) there exist at most (uniformly) boundedly many equivalence classes of the definable equivalence relation $E(p,q)$, for which the duo family of the constructed duo resolution, DuoRes, that is associated with the specialization of the edge groups, admit a test sequence that restrict to proofs that the couples (p_n,q_n) are in the equivalence class.

Proof: Identical to the proof of lemma 3.2.

□

The auxiliary resolutions that are associated with a duo resolution, $DuoRes$, that is constructed from a couple of multi-graded resolutions, $MGRes_u$ and $MGRes_v$, enable us to analyze those values of the parameters for which generic points of $DuoRes$ restrict to couples (p,q) that are in the equivalence relation $E(p,q)$.

To analyze equivalence classes of $E(p,q)$, for which there do not exist generic points of any Duo resolution, $DuoRes$, that restrict to specializations (p,q) in these classes, we need further constructions (for collecting non-generic specializations). As we did in proving theorem 3.1, with each Duo limit group, $DuoRes$, that is composed from a couple of resolutions, $MGRes_u$ and $MGRes_v$, we further associate finitely many extra limit groups, that we denote, $Exlim$.

We start by looking at all the specializations of (the completion of) $MGRes_u$ for which there exists a test sequence of specializations of (the completion of) $MGRes_v$, so that for each specialization in the combined sequence there exist extra rigid or (families of) strictly solid specializations (of one of the terminal limit groups $Term_1,\ldots,Term_s$ or one of the terminal rigid or solid limit groups of the Non-Rigid, Non-Solid, Left, Root, Extra PS, or Generic Collapse Extra PS resolutions that are associated with them) that are not specified by the corresponding specialization of the subgroup, $< f,p,q >$. Note that there is a global bound on the number of such (distinct) extra rigid or families of strictly solid specializations. By the techniques for constructing formal and graded formal limit groups (sections 2 and 3 in [Se2]), this collection of specializations can be collected in finitely many limit groups, and each has the form of a graph of groups in which a closure of $MGRes_v$ stabilizes a vertex group, and $MGRes_u$ can be mapped into another vertex group. Similarly, we look at the specializations of $MGRes_v$ for which there exists a test sequence of specializations of $MGRes_u$ so that the combined specializations have similar properties.

With an extra limit group, $Exlim$, we can naturally associate finitely many subgroups that are associated with the finite collection of extra rigid and extra solid specializations that are collected in the construction of $Exlim$. Each of these subgroups inherits a graph of groups decomposition from the graph of groups decomposition of $Exlim$. In this graph of groups decomposition there is one vertex group that contains the subgroup, $< q >$, and another vertex group that contains the subgroup $< p >$.

(1) The first vertex group is a subgroup of a closure of $MGRes_v$ (resp. $MGRes_u$), and is (multi-graded) rigid or solid with respect to the subgroup that is generated by $< q >$ (resp. $< p >$) and the edge groups that are connected to the vertex that is stabilized by $< q >$ (resp. $< p >$) in Λ. The second is a subgroup of the vertex group that is stabilized by $MGRes_u$ (resp. $MGRes_v$), and is (multi-graded) rigid or solid with respect to the subgroup that is generated by $< p >$ (resp. $< q >$) and the edge groups that are connected to the vertex group that is stabilized by $< p >$ (resp. $< q >$) in Λ.

(2) each extra rigid or strictly solid specialization that is collected by $Exlim$ restricts to (multi-graded) rigid or strictly solid specializations of a pair of subgroups that are described in part (1)..

We continue with all the (finitely many) Extra resolutions and Extra limit groups of the prescribed structure, that were constructed from the duo resolution, $DuoRes$, i.e., from the couple of resolutions, $MGRes_u$ and $MGRes_v$. As in the quantifier elimination procedure (the sieve procedure), for each Extra resolution, and Extra limit group, that are associated with $DuoRes$ (which is in particular a taut resolution), we collect all the specializations that factor and are taut with respect to the (taut) Duo resolution, $DuoRes$, and extend to specializations of either a resolution, $Extra$, or an Extra limit group, $Exlim$, and for which the elements that are supposed to be extra rigid or strictly solid specializations and are specified by these specializations collapse. This means that the elements that are supposed to be an extra rigid or strictly solid specializations are either not rigid or not strictly solid, or they coincide with a rigid specialization that is specified by the corresponding specialization of $< f, p, q >$, or they belong to a strictly solid family that is specified by $< f, p, q >$. These conditions on the elements that are supposed to be extra rigid or strictly solid specializations are clearly Diophantine conditions, hence, we can add elements that will demonstrate that the Diophantine conditions hold (see section 1 and 3 of [Se5] for more detailed explanation of these Diophantine conditions, and the way that they are imposed). By our standard methods (section 5 in [Se1]), with the entire collection of specializations that factor through an Extra resolution or an Extra limit group, and restrict to elements that are taut with respect to the (taut) duo resolution, $DuoRes$, and for which the elements that are supposed to be extra rigid or strictly solid specializations satisfy one of the finitely many possible (collapse) Diophantine conditions, together with specializations of elements that demonstrate the fulfillment of these Diophantine conditions, we can associate finitely many limit groups. We denote these limit groups, $ExCollape_1, \ldots, ExCollapse_g$, and call them *Extra Collapse* limit groups.

The Diophantine conditions, that are imposed on specializations of Extra resolutions and limit groups, require additional elements to be expressed. These additional elements do not generally obey the separation of variables that was achieved so far, i.e., it involves the entire duo resolution, $DuoRes$, that was constructed from the multi-graded resolutions, $MGRes_u$ and $MGRes_v$, and can not be imposed on each of its graded completions separately. To preserve the separation of variables, i.e., to divide the Diophantine condition into two Diophantine conditions that are imposed on the two graded completions "separately", we need to apply once again the procedure for separation of variables, that was presented and used in the previous section.

As we did in the previous section, to initiate the separation of variables procedure, we need to exclude free products. Let $DuoRes$ be a duo resolution, that is composed from the two multi-graded resolutions, $MGRes_u$ and $MGRes_v$. Let $< z, p, q >$ be the (duo) completion of $DuoRes$. Note that the subgroup $< f, p, q >$ is mapped into $< z, p, q >$.

We start by collecting all the specializations of the group $< z, p, q >$, for which:

(1) the restriction to the (image of the) subgroup $< f, p, q >$ form a proof that the couple (p, q) is in $E(p, q)$.
(2) the restriction to the specialization of the subgroup $< f, p, q >$ does not

factor through a free product of limit groups in which the subgroup $< p >$ is in one factor and the subgroup $< q >$ is contained in a second factor, and not through a free product in which the subgroup $< p, q >$ is contained in a factor.

(3) the ambient specialization (of the group $< z, p, q >$) factors through a free product of limit groups in which the subgroup $< p, q >$ (and hence, $< f, p, q >$) is contained in one factor, and either the subgroup that is associated with the completion of $MGRes_u$, or the one that is associated with the completion of $MGRes_v$, is not contained in one factor.

By our standard techniques (section 5 in [Se1]), the collection of all these specializations factor through finitely many limit groups. By looking at the actions of the ambient limit group $< z, p, q >$ on the Bass-Serre trees that are associated with the free products of limit groups through which the specializations factor, and apply the shortening procedure for these actions, by using only the modular groups of the subgroups that are associated with the completions of $MGRes_u$ and $MGRes_v$ relative to the subgroup $< f, p, q >$, we can replace these limit groups, by a collection of finitely many (quotient) limit groups, $GF_1, \ldots, GF_f$, so that each of them admits a free product in which the subgroup $< f, p, q >$ is contained in one factor, and either the completion of $MGRes_u$, or that of $MGRes_v$, is not mapped into a factor. Therefore, we can replace the collection of these specializations by the limit groups that are associated with the factors that contain the subgroup $< f, p, q >$ in each of the limit groups, $GF_1, \ldots, GF_f$, and this impose non-trivial relations on either the subgroup that is associated with the completion of $MGRes_u$, or the subgroup that is associated with the completion of $MGRes_v$.

We continue by collecting all the specializations of the Extra Collapse limit groups, $ExCollapse$, for which:

(1) the restriction to the (image of the) subgroup $< f, p, q >$ form a proof that the couple (p, q) is in $E(p, q)$.

(2) the restriction to the specialization of the subgroup $< f, p, q >$ does not factor through a free product of limit groups in which the subgroup $< p >$ is in one factor and the subgroup $< q >$ is contained in a second factor, and does not factor through a free product of limit groups in which the subgroup $< p, q >$ is contained in a factor.

(3) the restriction of the specialization of the Extra Collapse limit group, $ExCollapse$, to the subgroup $< z, p, q >$, that is associated with the duo resolution, $DuoRes$, from which it was constructed, does not factor through a free product of limit groups in which $< p, q >$ is in one factor.

(4) the ambient specialization (of $ExCollapse$) factors through a free product of limit groups in which the subgroup $< p, q >$ (and hence, $< z, p, q >$) is contained in one factor.

Once again, the collection of all these specializations (of $ExCollapse$) factor through finitely many limit groups, and by looking at the actions of these limit groups on Bass-Serre trees corresponding to the free products of limit groups through which the specializations factor, and apply the shortening procedure for these actions, by using the modular groups of the ambient group modulo the group $< z, p, q >$, we can replace these limit groups, by a collection of finitely many (quotient) limit groups, $AGF_1, \ldots, AGF_a$, so that each of them admits a free product in which the subgroup $< z, p, q >$ is contained in one factor. Therefore, we can

replace the collection of these specializations of the Extra Collapse limit groups, $ExCollapse$, by the limit groups that are associated with the factors that contain the subgroup $< z, p, q >$ in each of the limit groups, $AGF_1, \dots, AGF_a$, and these factors still demonstrate that the Diophantine condition, that is associated with the collapsing form, does hold.

The parameters that are associated with the uniformization limit group, $Unif$, that was constructed in the previous section, admit only boundedly many values for each equivalence class (up to the imaginaries that were presented in section 2), but they are not guaranteed to separate between equivalence classes. With the uniformization limit group, $Unif$, there is an associated subgroup, $< f, p, q >$, that inherits a graph of groups decomposition, Λ, from the graph of groups decomposition, Δ, of the ambient uniformization limit group, $Unif$. With Λ we associated finitely many multi-graded resolutions, $MGRes_u$ and $MGRes_v$, and with each pair of these resolutions, we associated finitely many duo resolutions, $DuoRes$. For each value of the parameters that are associated with the graph of groups Λ, there exist at most (uniformly) boundedly many duo families of specializations of the duo resolutions, $DuoRes$, for which generic points in these families restrict to couples, (p, q), that are in the equivalence relation, $E(p, q)$. However, in general there will exist equivalence classes of the equivalence relation, $E(p, q)$, with which we can not associate any (generic point of a) duo family of any of the duo limit groups, $DuoRes$.

For these classes we started to construct tools that will assist us in collecting non-generic specializations (in the duo families). Using these tools, the uniformization limit group, $Unif$, its subgroup, $< f, p, q >$, and its abelian decomposition, Λ, we iteratively construct a new collection of uniformization limit groups, that we call *Collapse uniformization limit groups*. These limit groups will enable us to impose further and further Diophantine conditions on the multi-graded resolutions, $MGRes_u$ and $MGRes_v$, from which the duo resolution, $DuoRes$, was constructed.

The Collapse uniformization limit groups allow us to impose the new Diophantine conditions on the two multi-graded resolutions separately, hence, we are able to run the sieve procedure [Se6] for the two resolutions, $MGREs_u$ and $MGRes_v$, separately. The procedure terminates by the termination of the sieve procedure (theorem 22 in [Se6]), and when it terminates it is guaranteed that for any given equivalence class, there exist a duo family of one of the Duo limit groups that are constructed along the iterative procedure, so that generic points in the duo family restrict to (a sequence of) specializations of the pair, (p, q), that are in the given equivalence class. The parameters that are associated with such duo families are (definable) class multi-functions, and by construction they do separate between classes. Hence, we finally obtain geometric elimination of imaginaries.

To analyze the (non-generic) pairs $(p, q) \in E(p, q)$, that extend to a specialization of the subgroup $< f, p, q >$, which is a valid proof, and this proof extends to a specialization of one of the duo limit groups, $DuoRes$, we start with the following.

We collect all the specializations of the group $< z, p, q >$ that factor through one of the limit groups, $GF_1, \dots, GF_f$. Note that the specializations that factor through each of these limit groups satisfy a non-trivial relation that is imposed on either the limit group that is associated with $MGRes_u$ or with $MGRes_v$ or with both of them. Therefore, we can proceed to the next step by applying the sieve procedure (that was presented in [Se6] for quantifier elimination) to the proper quotients of

the (multi-graded) completions of the resolutions $MGRes_u$ and $MGRes_v$. In this case the separation of variables is "built in", i.e., non-trivial relations are imposed on $MGRes_u$ or $MGRes_v$ separately.

We continue by looking at the collection of specializations so that their restrictions to the subgroups $< z, p, q >$ does not factor through a free product in which the subgroup $< p, q >$ is contained in a factor. In this case the Diophantine condition that is imposed on these specializations can not apriory be "separated" into two Diophantine conditions that are imposed separately on the completions of $MGRes_u$ and $MGRes_v$. Hence, we apply a modification of the procedure for separation of variables, to "separate" the imposed Diophantine condition to two separate Diophantine conditions that are imposed on $MGRes_u$ and $MGRes_v$.

The uniformization limit group, $Unif$, is composed from two graded completions, one that contains the subgroup $< p >$, and one that contains the subgroup $< q >$ (these are the two vertex groups in the graph of groups Δ that is associated with $Unif$, the one that contains $< p >$, and the one that contains $< q >$). We denote these two graded completions, $GComp_p$ and $GComp_q$.

We start with $GComp_p$, that is part of $Unif$, and the finitely many resolutions, $MGRes_u$. We look at all the test sequences of the completion $GComp_p$, that extend to (shortest) specializations that factor through and are taut with respect to one of the resolutions, $MGRes_u$, and do not factor through a free product of limit groups, in which the subgroup $< p, q >$ can be conjugated into a factor. By the techniques for constructing formal limit groups, that are presented in sections 2 and 3 in [Se2], with the collection of all such sequences we can associate finitely many graded completions, $Comp_1, \ldots, Comp_\ell$. The graded completion $GComp_p$ is naturally mapped into each of these completions, preserving the level structure, and each of the constructed completions terminate in either a rigid or a solid limit group, with respect to the parameter subgroup of the original graded completion, $GComp_p$. Furthermore, the terminal rigid or solid limit groups of $GComp_p$ and of $MGRes_u$, are mapped into the terminal rigid or solid limit group of the constructed completions, $Comp_1, \ldots, Comp_\ell$.

For presentation purposes we will assume that the graded abelian decompositions that are associated with the various levels of the completions, $Comp_1, \ldots, Comp_\ell$, contain no abelian vertex groups. In that case, given one of the completions, $Comp_j$, and a multi-graded resolution, $MGRes_u$, that is mapped into it, the image of the multi-graded resolution, $MGRes_u$, in the completion, $Comp_j$, intersects the conjugates of rigid vertex groups in the multi-graded abelian decomposition of the terminal rigid or solid limit group of $Comp_j$, in finitely many conjugacy classes of subgroups.

By the uniform bounds on the number of rigid and families of strictly solid specializations (theorems 2.5 and 2.9 in [Se3]), given a value of the parameters of the completion, $Comp_j$, which are the parameters of the uniformization limit group, $Unif$ (as well as its associated completions, $Comp_p$ and $Comp_q$)), the subgroups which are the intersections between the images of $MGRes_u$ in $Comp_j$, and conjugates of the rigid vertex groups in the multi-graded abelian decomposition that is associated with the terminal rigid or solid limit group in $Comp_j$, admit only (uniformly) boundedly many values (up to conjugation) for each possible value of the defining parameters. However, it is not guaranteed that these subgroups admit only boundedly many values (up to conjugacy) for each equivalence class of $E(p, q)$.

At this stage we use the construction of uniformization limit groups, that was

presented in the previous section. Starting with each of the completions, $Comp_j$, for which there exists an equivalence class, so that the subgroups of intersection between the image of $MGRes_u$ and conjugates of the rigid vertex groups in the abelian decomposition that is associated with the terminal level of $Comp_j$, admit infinitely many values (up to conjugacy) that are associated with the equivalence class, we do the following. We iteratively associate uniformization limit groups with the completion, $Comp_j$, and the collection of equivalence classes for which the set of associated parameters is infinite. By theorems 3.9 and 3.17, this iterative procedure for constructing uniformization limit groups, terminates after finitely many steps.

When this iterative procedure terminates, there are finitely many completions that are constructed along it. We denote these completions, $UComp_1, \ldots, UComp_t$. With each completion, $UComp_i$, there is a map from one of the (completions of the) multi-graded resolutions, $MGRes_u$, that is mapped into it. For presentation purposes we further assume that the abelian decompositions that are associated with the various levels of the constructed completions, $UComp_1, \ldots, UComp_t$, contain no abelian vertex groups. The image of $MGRes_u$ in such a completion, $UComp_i$, intersects conjugates of the rigid vertex groups in the terminal multi-graded abelian decompositions of the completion, $UComp_j$, in a finite collection of conjugacy classes of subgroups. The iterative procedure for separation of variables (i.e., for the construction of uniformization limit groups), that was presented in the previous section, guarantees that for each equivalence class of $E(p,q)$, for which there exists a duo family of $Unif$, and a test sequence of the duo family, that restricts to specializations of the subgroup $< f,p,q >$, that form valid proofs that the specializations of the couple, (p,q), are in the equivalence class, there exists at least one of the constructed completions, $UComp_i$, for which the values of the subgroups of intersection between the image of $MGRes_u$ and conjugates of the rigid vertex groups in the abelian decomposition of the terminal limit groups of the completion, $UComp_i$, admit only (uniformly) bounded number of values (up to conjugacy) that are associated with the given equivalence class.

In a similar way, we apply the procedure for the construction of uniformization limit groups to the set of completions that are associated with the graded completion, $GComp_q$, that is associated with the uniformization limit group, $Unif$. We denote the constructed graded completions, $VComp|_1, \ldots, VComp_r$. In that case one of the resolutions, $MGRes_v$ is mapped into each of the constructed completions. The constructed completions have similar properties with respect to the subgroups of intersection with the image of $MGRes_v$, as the completions that are constructed from $GComp_p$ have with respect to subgroups of intersection with the image of $MGRes_u$.

The construction of the completions, $UComp_1, \ldots, UComp_t$, and $VComp_1, \ldots, VComp_r$, associates a universal and canonical finite set of completions with the uniformization limit group, $Unif$, so that into each of the completions, $UComp_j$, there is a map of (the completion of) one of the multi-graded resolution, $MGRes_u$, and into each completion, $VComp_i$, there is a map of (the completion of) one of the multi-graded resolutions, $MGRes_v$. The completions, $UComp_j$ and $VComp_i$, enable one to extend specializations from the subgroup $< f,p,q >$ of the original limit group, $Unif$, to couples of duo resolutions, $MGRes_u$ and $MGRes_v$, i.e., to the duo resolutions, $DuoRes$. However, it is still not sufficient for extending specializations of the subgroup, $< f,p,q >$, or the duo resolution, $DuoRes$, to the

extra collapse limit groups, $Excollapse_1, \dots, ExCollapse_g$, in a way that separates variables (as we did in the previous section). Such a separation that will allow us to impose new Diophantine conditions on the multi-graded resolutions, $MGRes_u$ and $MGRes_v$, separately, and hence enable us to continue to the second step of a sieve procedure [Se6], that will associate new (developing) resolutions with $MGRes_u$ and $MGREs_v$.

To extend (generic) specializations of the uniformization limit group, $Unif$, and their restriction to the subgroup, $< f, p, q >$, to the Extra Collapse limit groups, $ExCollapse_1, \dots, ExCollapse_g$, we once again apply the procedure for separation of variables, i.e., we iteratively associate with each couple of completions, $UComp_j$ and $VComp_i$, a (finite) sequence of uniformization limit groups.

We start with all the possible couples of completions, $UComp_j$ and $VComp_i$, in parallel. Hence, for brevity we denote such a couple, $UComp$ and $VComp$. We look at all the duo families of the graded completion, $UComp$, that admit a test sequence of specializations, so that this test sequence of specializations have the following properties:

(1) the test sequence of specializations of $UComp$ can be extended to a sequence of specializations, that is composed from a test sequence of specializations of both graded completions, $UComp$ and $VComp$, so that the restrictions of these specializations to the subgroup, $< f, p, q >$, form valid proofs that the specializations of the pair (p, q), are in the definable equivalence relation, $E(p, q)$.

(2) the sequence of specializations that is composed from test sequences of $UComp$ and $VComp$, extends to (shortest) specializations of either one of the extra Collapse limit groups, $ExCollapse_1, \dots, ExCollapse_g$, or one of their associated quotients, $AGF_1, \dots, AGF_a$, so that these specializations do not factor through a free product in which the subgroup, $< p, q >$, is contained in a factor.

By the techniques of the previous section, and the techniques for constructing formal graded limit groups that were presented in sections 2 and 3 in [Se2], with the collection of sequences of specializations, that extend test sequences of $UComp$, and satisfy properties (1) and(2), it is possible to associate finitely many duo limit groups, that we call $Collapse\ uniformization\ limitgroups$, and denote $ColUnif$.

Every collapse uniformization limit group is in particular a duo limit group, so with it we can associate two graded completions. $UComp$ is mapped into one of the two graded completions that is associated with $ColUnif$, using a natural map that preserves the level structure of $UComp$. In fact, this graded completion of $ColUnif$ has the same structure as that of $UComp$, except for the terminal rigid or solid limit group of $UComp$ and $ColUnif$.

$VComp$ is mapped into the second completion that is associated with $ColUnif$, by a map that preserve the level structures of $VComp$, except (perhaps) for the terminal level of $VComp$.

The subgroup $< f, p, q >$ is mapped into $ColUnif$, and generic points in $ColUnif$ restrict to specializations of $< f, p, q >$ that form valid proofs that the specializations of the pair (p, q) is in the given definable equivalence relation, $E(p, q)$. By construction, the images of $UComp$ and $VComp$, restrict to images of a pair of multi-graded resolutions, $MGRes_u$ and $MGRes_v$, that are associated with $UComp$ and $VComp$, and hence to an image of the duo resolution that is constructed from

$MGRes_u$ and $MGRes_v$, that we denote, $DuoRes$. Finally, by property (2) of the specializations from which $ColUnif$ is constructed, the image of $DuoRes$ in $ColUnif$ extends to an image of one of the extra collapse limit groups, $ExCollapse_1, \dots, ExCollapse_g$ or one of their associated quotients, $AGF_1, \dots, AGF_a$.

For presentation purposes we will assume that the abelian decompositions that are associated with the various levels of the constructed collapse uniformization limit groups, $ColUnif$, contain no non-cyclic abelian vertex groups.

With each collapse uniformization limit group, $ColUnif$, there is an associated map from an extra collapse limit group, $ExCollapse$, or one of its associated quotients, $AGF_1, \dots, AGF_a$, into it. The image of the extra collapse limit group, $ExCollapse$, in $ColUnif$, does not factor as a free product in which $< p >$ is contained in one factor and $< q >$ in another factor, and it does not factor as a free product in which $< p, q >$ is contained in a factor. Hence, by proposition 3.3, the image of $ExCollapse$ in $ColUnif$ intersects the rigid vertex groups in the abelian decomposition that is associated with the terminal limit group of $ColUnif$ in conjugates of finitely many (f.g.) subgroups.

By proposition 3.4, for each given value of the defining parameters of $ColUnif$, the number of specializations of these intersection subgroups (up to conjugacy) is uniformly bounded. However, there may be equivalence classes of the definable equivalence relation, $E(p,q)$, for which there are infinitely many conjugacy classes of values of the subgroups of intersection between the image of $ExCollapse$ and the rigid vertex groups in the abelian decomposition that is associated with the terminal limit group of $ColUnif$.

In this case, we continue by iteratively construct (collapse) uniformization limit groups, in a similar way to what we did in the procedure for separation variables in the previous section. For presentation purposes we assume that the abelian decompositions that are associated with the various levels of the constructed collapse uniformization limit groups contain no abelian vertex groups. By theorem 3.9, this iterative procedure for the construction of collapse uniformization limit groups terminates after finitely many steps.

When the iterative procedure terminates we are left with finitely many collapse uniformization limit groups, that we denote, $ColUnif_1, \dots, ColUnif_b$. With each such collapse uniformization limit group there is an associated map from one of the extra collapse limit groups, $ExCollapse$, and one of the duo resolutions, $DuoRes$, that is composed from a pair of resolutions, $MGRes_u$ and $MGRes_v$. Furthermore, by theorem 3.11, for each equivalence class of $E(p,q)$, which is not:

(i) one of the finitely many equivalence classes that were singled out in theorem 3.1

(ii) one of the equivalence classes for which there exists a (duo) test sequence of one of the duo resolutions, $DuoRes$, that restricts to valid proofs that the specializations of the pair, (p,q), are in the equivalence class (see lemma 4.1).

and for which there are duo families of one of the uniformization limit groups that were constructed in the previous section (the uniformization limit groups, $Unif_1, \dots, Unif_v$), so that generic specializations in these duo families restrict to valid proofs that the specializations of the pairs, (p,q), are in the equivalence class, there exists a collapse uniformization limit group, $ColUnif$, with the following properties:

(1) there are duo families of $ColUnif$, for which generic points in these duo families restrict to valid proofs that the specializations of the pair, (p,q), are in the equivalence class.
(2) the subgroups of intersection between conjugates of the rigid vertex groups in the abelian decomposition that is associated with the terminal rigid or solid limit group of $ColUnif$, and the image of *ExCollapse* in $ColUnif$, admit only uniformly bounded number of conjugacy classes of specializations that are associated with the given equivalence class.

With each collapse limit group, $ColUnif_1,\dots,ColUnif_b$, we associate the collection of equivalence classes of $E(p,q)$, that are not one of the equivalence classes that satisfy (i) or (ii), and for which properties (1) and (2) hold for them and for the collapse uniformization limit group. We continue with the (finitely many) collapse uniformization limit groups in parallel, hence, we denote the one we continue with, $ColUnif$.

With the collapse uniformization limit group, $ColUnif$, we have associated maps from a pair of multi-graded resolutions, $MGRes_u$ and $MGRes_v$, that extend to a map from the duo resolution that is composed from the two multi-graded resolution, that we denoted, *DuoRes*. Furthermore, the map from *DuoRes*, extends to a map from an extra collapse limit group, *ExCollapse*, that is associated with *DuoRes*.

In the previous section we explained how the subgroup, $< f,p,q >$, inherits a graph of groups decomposition from each of the uniformization limit groups, $Unif_1,\dots,Unif_v$ (see lemmas 3.12 and 3.14). The collapse uniformization limit group, $ColUnif$, being a duo limit group, admits a graph of groups decomposition. Like the subgroup $< f,p,q >$ in a uniformization limit group, $Unif$, the image of an extra collapse limit group, *ExCollapse*, inherits a graph of groups decomposition from the graph of groups decomposition that is associated with $ColUnif$ (as a duo limit group).

Let Λ_{Cl}^{Ex} be the graph of groups that is inherited by the image of *ExCollapse* from $ColUnif$. Λ_{Cl}^{Ex} have similar properties to those of the graph of groups, Λ.

Lemma 4.2. *Suppose that the terminal limit groups in all the collapse uniformization limit groups that were constructed along our iterative procedure are rigid or solid, and the graded completions that are associated with them contain no abelian vertex groups in any of their levels. Let Λ_{Cl}^{Ex} be the graph of groups decomposition that is inherited by the image of the extra collapse limit group, ExCollapse, from the collapse uniformization limit group, $ColUnif$. Then Λ_{Cl}^{Ex} is either:*

(1) *Λ_{Cl}^{Ex} is a trivial graph, i.e., a graph that contains a single vertex. In that case either the subgroup $< p >$ or the subgroup $< q >$ is contained in the distinguished vertex group in the abelian decomposition that is associated with the distinguished vertex group in $ColUnif$, i.e., the vertex group that contains the coefficient group. In particular, either the subgroup $< p >$ or $< q >$ admits boundedly many values.*
(2) *Λ_{Cl}^{Ex} has more than one vertex group, and both subgroups $< p >$ and $< q >$ are contained in the same vertex group. Like in case (1), in that case either the subgroup $< p >$ or the subgroup $< q >$ admit only boundedly many values.*
(3) *Λ_{Cl}^{Ex} has more than one vertex, $< p >$ is contained in the stabilizer of one vertex and $< q >$ is contained in the stabilizer of another vertex in Λ_i.*

In that case the image of the (completion of the) multi-graded resolution $MGRes_u$ is contained in the vertex group that contains $< p >$, and the image of the (completion of the) multi-graded resolution $MGRes_v$ is contained in the vertex group that contains $< q >$. Furthermore, given an equivalence class of $E(p,q)$ that is associated with $ColUnif$, the number of conjugacy classes of the values of edge groups in Λ_{Cl}^{Ex} that are associated with the given equivalence class, are uniformly bounded.

Proof: Identical to the proofs of lemmas 3.12 and 3.14.

□

The graph of groups, Λ_{Cl}^{Ex}, of the image of the extra collapse limit group, $ExCollapse$, and its properties (lemma 4.2), enable us to continue to the next step of the sieve procedure [Se6]. By lemma 4.2, the subgroup $< p >$ is contained in one vertex group in Λ_{Cl}^{Ex}, that contains the image of the completion of $MGRes_u$, and the subgroup $< q >$ is contained in a second vertex group in Λ_{Cl}^{Ex}, that contains the image of the completion of $MGRes_v$. The edge groups in Λ_{Cl}^{Ex} admit only (uniformly) boundedly many values (up to conjugacy) that are associated with each equivalence class of $E(p,q)$ that is associated with the collapse uniformization limit group, $ColUnif$. Therefore, the graph of groups, Λ_{Cl}^{Ex}, enables us to separate the Diophantine conditions that are imposed by the extra collapse limit group, $ExCollapse$, into two Diophantine conditions that are imposed separately on the two taut multi-graded resolutions, $MGRes_u$ and $MGRes_v$.

Given the two separate Diophantine conditions that are imposed on the two multi-graded resolutions, $MGRes_u$ and $MGRes_v$, we analyze their quotient resolutions according to the procedure for the analysis of quotient resolutions that is presented in the second step of the sieve procedure in [Se6]. This procedure associates with $MGRes_u$ and with $MGRes_v$, and the Diophantine conditions that are imposed on them, finitely many quotient resolutions, and with each quotient resolution it associates a (finite) sequence of core resolutions, a developing resolution, and possibly a sculpted resolution (see [Se6] for the definition of these notions and for their construction).

Given each possible couple of quotient multi-graded resolutions, one that was obtained from $MGRes_u$ and a second that was obtained from $MGRes_v$, we first look at their associated developing resolutions. Given the pair of their associated developing resolution, we construct from the pair a duo resolution, that we denote, $DuoDevRes$. With a duo resolution, $DuoDevRes$, we associate a finite collection of auxiliary resolutions, in a similar way to the auxiliary resolutions that were constructed along the sieve procedure [Se6], and the auxiliary resolutions that were associated with the duo resolution, $DuoRes$, in the first step of our iterative procedure. These include, Non-rigid, Non-solid, Root, Left, Extra, and Generic Collapse Extra resolutions (note that all the auxiliary resolutions are closures of the duo resolution, $DuoDevRes$).

Given a duo resolution, $DuoDevRes$, that is composed from a pair of (completions of) developing resolutions, and its auxiliary resolution, we first separate those equivalence classes of $E(p,q)$, for which there exists a (duo) test sequence of $DuoDevRes$, that restricts to valid proofs that the specializations of the pair, (p,q), are in the equivalence class.

Given each Extra resolution of $DuoDevRes$, we first collect those specializations that factor through a free product in which the subgroup, $< p,q >$, is contained

in a factor. With these specializations we associate finitely many limit groups, in a similar way to the construction of the limit groups, $GF_1,\dots,GF_f$, that we used in the first step of the iterative procedure. Each of the constructed limit groups imposed a non-trivial relation on at least one of its associated developing resolutions, hence, we continue with it to the next step of the sieve procedure. Then, with an Extra resolution, we associate finitely many Extra Collapse limit groups (as in the first step of the procedure). We further separate all these specializations of an Extra Collapse limit group that factor through a free product of limit groups in which the subgroup $<p,q>$ is contained in a factor. Hence, we may assume that the specializations of the Extra Collapse limit groups that we consider do not factor through such a free product.

Given the finite collection of quotient resolutions, and their associated developing resolutions, Extra resolutions, and Extra Collapse limit groups, we use the iterative procedure for separation of variables, that was used in the first step, to associate with this collection a finite collection of Collapse uniformization limit groups. With each collapse uniformization limit group, there is an associated map from a duo resolution, $DevDuoRes$, that is composed from a pair of developing resolutions, that extends to maps from the two associated quotient resolutions. The map from the duo resolution, $DevDuoRes$, extends also to a map from an associated Extra resolution, and this extends to a map from a Collapse Extra limit group.

With the subgroup that is generated by the images of the two quotient resolutions, and by the image of the Collapse Extra resolution, that are associated with a Collapse uniformization limit group, we associate a graph of groups decomposition that we (also) denote, Λ^{Ex}_{Cl}. This graph of groups have similar properties to the ones listed in lemma 4.2. Each of the quotient resolutions that are associated with the Collapse uniformization limit group is mapped to a vertex group, one to a vertex group that contains $<p>$, and the second to a vertex group that contains $<q>$.

As in the first step, the graph of groups, Λ^{Ex}_{Cl}, gives a separation of the Diophantine condition that is imposed by the Collapse Extra limit group on the specializations that factor and are taut with respect to the Extra resolution, into two separate Diophantine conditions that are imposed on the two quotient resolutions that are mapped into the Collapse uniformization limit group.
Once the Diophantine condition is separated, we can apply the general step of the sieve procedure, and associate with a Collapse uniformization limit group, its associated graph of groups, Λ^{Ex}_{Cl}, and its associated Collapse Extra limit groups and pair of quotient resolutions, a finite collection of (new) quotient resolutions, their associated finite sequences of core resolutions, developing resolutions, and possibly sculpted resolutions (see [Se6] for these objects and their construction).

We continue iteratively. By the termination of the sieve procedure, the iterative procedure that combines the procedure for separation of variables, with the sieve procedure, terminates after finitely many steps.

Theorem 4.3. *In case there are no abelian vertex groups in any of the abelian decompositions that are associated with the various levels of the collapse uniformization limit groups that are constructed along the iterative procedure for the analysis of the parameters of equivalence classes, the iterative procedure, that combines the sieve procedure with the procedure for separation of variables, terminates after finitely many steps.*

Proof: Follows from the termination of the sieve procedure (theorem 22 in [Se6]). □

Theorem 4.3 guarantees the termination of the iterative procedure, that we used in order to analyze parameters that can be assigned with the equivalence classes of a given definable equivalence relation, $E(p,q)$, in case there are no abelian vertex groups in any of the abelian decompositions that are associated with the various levels of the constructed collapse uniformization limit groups.

Suppose that the abelian decompositions that are associated with the various levels of the uniformization limit groups, $Unif_1, \dots, Unif_v$, that were constructed in the previous section, do contain abelian vertex groups. In this general case, with each of the uniformization limit groups, $Unif_1, \dots, Unif_v$, that were constructed in the procedure for separation of variables in the previous section, there is (also) an associated subgroup, $< f, p, q >$, which is the image of one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$, that were constructed in theorem 3.1, and a graph of groups decomposition, (also denoted) Λ, that is described in lemma 3.18.

The graph of groups, Λ, gives us a separation of variables in the general case as well. Given Λ, we can associate the taut multi-graded abelian Makanin-Razborov diagram of the vertex group in Λ that contains the subgroup $< p >$, and the taut multi-graded Makanin-Razborov diagram of the vertex group in Λ that contains $< q >$.

We continue precisely as we did in case there were no abelian vertex groups in the constructed uniformization limit groups. With each pair of resolutions, $MGRes_u$, of the vertex group that contains $< p >$, and $MGRes_v$, of the vertex group that contains $< q >$, we associate a finite (possibly empty) collection of duo resolutions, *DuoRes*. Given a duo resolution, *DuoRes*, we associate with it, Non-rigid, Non-Solid, Left, Root, Extra, and Generic Collapse extra (duo) resolutions. With each Extra resolution, we further associate a finite collection of Collapse Extra limit groups. Given the duo resolutions, *DuoRes*, and their associated Extra Collapse limit groups, we apply the same iterative procedure that was used in the absence of abelian vertex groups, that terminates after finitely many steps according to theorem 3.17, and produces a finite collection of Collapse uniformization limit groups.

According to lemma 3.18, with each of the collapse uniformization limit groups, there is an associated graph of groups decomposition, Λ^{Ex}_{Cl}, of the image of the collapse extra limit group that is mapped into it. This graph of groups decomposition separates the Diophantine condition which is imposed by the collapsed extra limit group on the duo resolution, *DuoRes*, into two separate Diophantine conditions that are imposed on the two resolutions, $MGRes_u$ and $MGRes_v$, precisely as in case the associated abelian decompositions contained no abelian vertex groups. This separation of the Diophantine condition enable us to continue to the second step of the sieve procedure [Se6], and we continue iteratively until the termination of the sieve procedure that follows by theorem 4.3 (or rather by theorem 22 in [Se6]).

The iterative construction of the uniformization limit groups, $Unif_1, \dots, Unif_v$, in the previous section, and the iterative construction of developing resolutions (that are constructed according to the sieve procedure [Se6]), their auxiliary resolutions, in particular their associated Extra and Generic collapse extra resolutions, their associated Collapse Extra limit groups, and finally the iterative construction of the Collapse uniformization limit groups, enable one to associate parameters with

the various equivalence classes of the given definable equivalence relation, $E(p,q)$. The collection of these objects allows one to associate a collection of finitely many elements in finitely many limit groups, with the equivalence relation $E(p,q)$, and for each equivalence class in $E(p,q)$, these elements admit only (uniformly) boundedly many specializations, up to conjugation, left, right and double cosets of certain cyclic subgroups (cyclic subgroups of the coefficient group). The existence of these parameters follows from the construction we presented, in particular the properties of the collapse uniformization limit groups. To obtain geometric elimination of imaginaries, after adding sorts for the 3 basic (families of) imaginaries, we still need to show that the parameters we found are definable.

Theorem 4.4. *Let F be a (non-abelian) free group, and let $E(p,q)$ be a definable equivalence relation over F. If we add sorts for the imaginaries that are presented in section 2: conjugation, left and right cosets of cyclic groups, and double cosets of cyclic groups, then $E(p,q)$ is geometrically eliminated.*

Suppose that p and q are m-tuples. There exist some integers s and t and a definable multi-function:

$$f : F^m \ \to \ F^s \times R_1 \times \ldots \times R_t$$

where each of the R_i's is a new sort for one of the 3 basic imaginaries (conjugation, left, right and double cosets of cyclic groups). The image of an element is uniformly bounded (and can be assumed to be of equal size), the multi-function is a class function, i.e., two elements in an equivalence class of $E(p,q)$ have the same image, and the multi-function f separates between classes, i.e., the images of elements from distinct equivalence classes is distinct. Furthermore, if $E(p,q)$ is coefficient-free, then we can choose the definable multi-function f to be coefficient-free (although then, the image of the multi-function may be of different (bounded finite) cardinalities, for different classes).

Proof: Let $E(p,q)$ be a definable equivalence relation. To prove that $E(p,q)$ can be geometrically eliminated, we need to construct a definable multi-function f as described in the theorem. First, in theorem 3.1 we associated with $E(p,q)$ finitely many rigid or solid limit groups, $Ipr_1, \ldots, Ipr_w$, so that for all but finitely many equivalence classes, and for every pair (p,q) in any of the remaining equivalence classes, there is a rigid or a strictly solid homomorphism from one of these limit groups into the coefficient group F, that restricts to a proof that the pair (p,q) is in $E(p,q)$, and the homomorphism does not factor through a free product of limit groups in which $< p >$ is contained in one factor, and $< q >$ is contained in a second factor.
All our further constructions (of uniformization limit groups) are based on the existence of such homomorphisms, and hence, the finitely many equivalence classes that were singled out by theorem 3.1 are excluded. Therefore, to construct the desired multi-function f, we need to show that the finite collection of equivalence classes that were singled out in theorem 3.1, is a definable collection.

Proposition 4.5. *Let $E(p,q)$ be an equivalence class over a free group F. The finite collection of equivalence classes that were singled out in theorem 3.1, is a definable collection, i.e., if the equivalence relation $E(p,q)$ is coefficient-free, then*

the finite collection of equivalence classes that are singled out in theorem 3.1, is coefficient-free definable.

Proof: Recall that in order to prove theorem 3.1 we have constructed finitely many limit groups, $GFD_1, \ldots, GFD_d$, that admit a free product decomposition in which $< p >$ is contained in one factor, and $< q >$ is contained in a second factor. With each limit group GFD we have associated its taut Makanin-Razborov diagram. The finitely many equivalence classes of $E(p,q)$ that were excluded in theorem 3.1, are precisely those equivalence classes for which there exists a test sequence of one of the (finitely many) resolutions in these diagrams that restrict to valid proofs that the specializations of the pair, (p,q), are in the equivalence class. By lemma 3.2 there are only finitely many such equivalence classes.

Let Res be one of the resolutions in the taut Makanin-Razborov diagrams of the limit groups, $GFD_1, \ldots, GFD_d$. With the resolution, Res, we have associated (along the proof of theorem 3.1) finitely many auxiliary resolutions. These include Non-rigid, Non-solid, Left, Root, Extra and Generic Collapse resolutions. With each Extra resolution, we have further associated Extra limit groups, that we denoted $Exlim$, and each Extra limit group admit a free product, $Exlim = Exlim_1 * Exlim_2$. With each Extra limit group we associated finitely many (possibly none) Collapse Extra limit groups.

With an Extra limit group, $Exlim$, we further associate a finite collection of limit groups, that we call *Generic Collapse Extra* limit groups. Recall that a limit group, GFD_i admits a free product in which $< p >$ is in one factor, and $< q >$ is in a second factor. Hence, a resolution, Res, in a Makanin-Razborov diagram of GFD factors as a free product of resolutions, Res_1 and Res_2, where $< p >$ is a subgroup of the completion of Res_1, and $< q >$ is a subgroup of the completion of Res_2. An Extra limit group, $Exlim$ admits a free product, $Exlim = Exlim_1 * Exlim_2$, where either $Exlim_1$ is a closure of (the completion of) Res_1, or $Exlim_2$ is a closure of (the completion of) Res_2.

Recall that if a specialization of an Extra limit group, $Exlim$, restricts to a valid proof that the specialization of the pair, (p,q), is in the equivalence relation, $E(p,q)$, then the specialization of $Exlim$ satisfies one of finitely many Diophantine (collapse) conditions. Let $Exlim = Exlim_1 * Exlim_2$, and suppose that in $Exlim$, $Exlim_1$ is a closure of the resolution, Res_1. We look at all the specializations of $Exlim_2$ that can be extended by a sequence of specializations of $Exlim_1$ that restrict to a test sequence of specializations of the resolution Res_1, so that the combined sequence of specializations satisfy one of the finitely many Diophantine conditions that are associated with the Diophantine (collapse) conditions on specializations of $Exlim$.

If we add new elements that demonstrate the fulfillment of the imposed Diophantine condition, and choose these additional elements to have values that are the shortest possible, then by the techniques of collecting formal solutions (sections 2 and 3 in [Se2]), the collection of all such sequences of specializations of $Exlim$ that satisfy the collapse conditions, extended to (shortest) specializations that demonstrate the fulfillment of the (collapse) Diophantine conditions, factor through finitely many limit groups, that we call *Generic Collapse Extra* limit groups, $GenColExtra$. Each of these limit groups factors as a free product, $GenColExtra = GCE_1 * GCE_2$, and GCE_1 is a closure of Res_1 (and of $Exlim_1$). We repeat this construction for all the other Extra limit groups, $Exlim$, looking at sequences of specializations that restrict to test sequences of Res_2 in case $Exlim_2$

is a closure of Res_2.

The limit groups, $GFD_1, \dots, GFD_d$, that were constructed in the course of proving theorem 3.1, the (finitely many) resolutions that appear in their taut Makanin-Razborov diagrams, and their associated Non-rigid, Non-solid, Left, Root, Extra, and Generic Collapse Extra resolutions, together with the Extra limit groups, Collapse Extra limit groups, and finally the Generic Collapse Extra limit groups, that are associated with each of the finitely many Extra resolutions, enable us to construct a predicate that defines precisely the finitely many equivalence classes that are excluded in theorem 3.1.

Let p_0 be a specialization of the elements p. Recall that p_0 belongs to one of the equivalence classes that are singled out in theorem 3.1, if there exists a test sequence of one of the resolutions in the taut Makanin-Razborov diagrams of the limit groups, $GFD_1, \dots, GFD_d$, that restrict to valid proofs that the associated pairs (p_n, q_n) are in $E(p,q)$, and the elements, p_n and q_n, are in the same equivalence class as p_0. The existence of such a test sequence is equivalent to the existence of a specialization, (p_1, q_1), of the pair, (p,q), that satisfies the following conditions (that are all first order):

(1) $(p_1, q_1) \in E(p,q)$ and $(p_1, p_0) \in E(p,q)$ (i.e., p_1 and q_1 are in the same equivalence class as p_0).
(2) there exists a specialization, (p_1, q_1, f), which is a rigid or a strictly solid specialization of one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$, that restricts to a valid proof that $(p_1, q_1) \in E(p,q)$.
(3) the specialization, (p_1, q_1, f), extends to a specialization, (p_1, q_1, f, u), that factors through (the completion of) one of the resolutions, Res, which is a resolution in one of the taut Makanin-Razborov diagrams of one of the limit groups, $GFD_1, \dots, GFD_j$.
(4) (p_1, q_1, f, u) does not extend to a specialization that factors through any of the Non-rigid, Non-solid, Left, and Root resolutions that are associated with Res. If it extends to a specialization that factors through an Extra resolution that is associated with Res, the extended specialization must further extend to a specialization that factors through a Generic Collapse Extra resolution that is associated with the extra resolution Extra.
(5) If the specialization, (p_1, q_1, f, u), extends to a specialization of an Extra limit group, $Exlim$, then the (extended) specialization of $Exlim$, further extends to a specialization that factors through one of the Generic Collapse Extra limit groups that are associated with the Extra limit group, $Exlim$.

The construction of the auxiliary resolutions that are associated with a resolution, Res, in one of the Makanin-Razborov diagrams of the limit groups, $GFD_1, \dots, GFD_d$, guarantee that the existence of a test sequence of Res that restricts to valid proofs and specializations in a given equivalence class of $E(p,q)$ is equivalent to the existence of a pair (p_1, q_1) in the given equivalence class, that satisfies properties (1)-(5). It is fairly straightforward that properties (1)-(5), that are imposed on a pair, (p_1, q_1), are first order. Hence, the (finite) collection of equivalence classes that are singled out in theorem 3.1 is definable. Furthermore, in case the equivalence relation is coefficient-free, the limit groups, $GFD_1, \dots, GFD_d$, their Makanin-Razborov diagrams, and all the auxiliary resolutions and limit groups that are associated with the resolutions in these diagrams, are coefficient-free. Hence, the first order predicate that formulates properties (1)-(5) is coefficient-free. Therefore,

in case $E(p,q)$ is coefficient-free, the predicate that defines the finite collection of equivalence classes that are singled out in theorem 3.1 is coefficient-free.

$\square$

Proposition 4.5 shows that the finite collection of equivalence classes that are excluded in theorem 3.1 is definable. Therefore, to prove theorem 4.4, we need to construct a (definable) function with the properties that are listed in the statement of the theorem, that is defined on the union of all the other equivalence classes of $E(p,q)$.

Let p_0 be a specialization of the (free) variables, p, that does not belong to one of the finitely many equivalence classes that are excluded in theorem 3.1. Then for each pair $(p,q) \in E(p,q)$, that are in the same equivalence class as p_0, there exists a rigid or a strictly solid homomorphism h from one of the rigid or strictly solid limit group, $Ipr_1, \ldots, Ipr_w$, into the coefficient group F, that restricts to a valid proof that the given pair, (p,q) is in $E(p,q)$, and so that the homomorphism h, and all the homomorphisms in its strictly solid family, do not factor through a free product in which $< p >$ is contained in one factor, and $< q >$ is contained in a second factor.

Based on the existence of such a homomorphism, we have associated (in section 3) at least one uniformization limit group from the finite collection, $Unif_1, \ldots, Unif_v$, with the equivalence class of p_0, so that the uniformization limit group satisfies the conclusions of lemma 3.18 with respect to that equivalence class.

Furthermore, with the equivalence class of p_0, which is not one of the equivalence classes that are excluded in theorem 3.1, there is an associated Collapse uniformization limit group, $ColUnif$. With the graph of groups that is inherited by the (image of the) Collapse Extra limit group that is mapped into the collapse uniformization limit group, $ColUnif$, there is a pair of associated multi-graded (quotient) resolutions. A quotient multi-graded resolution that is associated with the vertex group of the vertex stabilized by $< p >$, and a quotient multi-graded resolution of the vertex group of the vertex stabilized by $< q >$. The pair of developing resolutions of these multi-graded resolutions, extends to a Duo resolution, that we denote, $DuoDevRes$, and there exists a (duo) test sequence, in a duo family of this duo resolution, that restricts to valid proofs that the corresponding specializations of the pair (p,q), (p_n, q_n), are in $E(p,q)$, and these specializations are in the same equivalence class as p_0.

To construct a predicate that defines a multi-function with the properties that are listed in the statement of theorem 4.4, i.e., to definably associate parameters with the equivalence classes of $E(p,q)$, we construct new duo limit groups that combine the (finitely many) Collapse Uniformization limit groups, $ColUnif$, with the (finitely many) duo resolutions that are associated with them, and are constructed from pairs of multi-graded developing resolutions, that are constructed from the graph of groups that is inherited by the image of the collapse extra limit groups in the corresponding collapse uniformization limit group. We call these new duo limit groups, *parameters duo limit groups*.

We construct the finitely many parameters duo limit groups for each of the (finitely many) Collapse uniformization limit groups in parallel. Recall that given a Collapse uniformization limit group, $ColUnif$, there exists a Collapse Extra limit group that is mapped into it. This Collapse Extra limit group inherits a graph of groups decomposition from the graph of groups that is associated with $ColUnif$, the subgroup $< p >$ is contained in a vertex group in this graph of groups, and the

subgroup $< q >$ is contained in another vertex groups in the graph of groups. With the vertex groups that contain the subgroups $< p >$ and $< q >$ we associated a finite collection of multi-graded quotient resolutions (by applying the general step of the sieve procedure), and with each such multi-graded resolution we associated its developing resolution. Given a pair of developing resolutions, one that is associated with the vertex group that contains $< p >$, and one that is associated with the vertex group that contains $< q >$, we constructed a duo resolution, that we called a developing duo resolution, $DevDuoRes$.

Given $ColUnif$, we construct finitely many (possibly none) parameters duo limit groups for each of its associated developing duo resolutions in parallel. Given $ColUnif$, and one of its (finitely many) developing duo resolutions, $DevDuoRes$, we look at the following sequences of specializations.

(1) we look at equivalence classes of the given equivalence relation, $E(p, q)$, for which the given set of elements of the ambient Collapse uniformization limit group, $ColUnif$, that are associated (and generate) the edge groups in the graph of groups that is inherited by the collapse extra limit group, that is mapped into $ColUnif$, admit only boundedly many values up to conjugation, and right, left and double cosets of the corresponding cyclic groups (see part (4) of lemma 3.18 for the elements that are associated with the edge groups in the inherited graph of groups).
(2) for each equivalence class that satisfies part (1), we look at a (duo) test sequence of $ColUnif$, that restricts to valid proofs that the sequence of pairs, (p_n, q_n), are in the equivalence class.
(3) we further require the (duo) test sequence of $Colunif$, that satisfies part (2), to extend to a (duo) test sequence of the associated developing duo resolution, $DevDuoRes$.
(4) Furthermore, we require that to the (duo) test sequence of $ColUnif$, and the (duo) test sequence of $DevDuoRes$, there exists an additional sequence of specializations of one of the rigid or solid limit groups, $Ipr_1, \dots, Ipr_w$, that demonstrate that the specializations of the pair, (p, q), in the two test sequences belong to the same equivalence class of $E(p, q)$.

By the techniques for the construction of formal solutions that appear in sections a and 3 in [Se2], with the collection of sequences that satisfy properties (1)-(4) it is possible to associate a finite collection of duo limit groups, that we call *parameters duo limit groups*. Note that each parameters duo limit group is composed from closures of the completions that appear in $ColUnif$ and $DevDuoRes$, that are associated with the parameters duo limit group.

As we did in the sieve procedure, with a parameters duo limit group we associate finitely many auxiliary resolutions. With each parameters duo limit group we associate a finite collection of Non-rigid, Non-solid, Left, Root, and Extra resolutions, that demonstrate the possible failure of the restriction of duo test sequences of the parameters duo limit group to sequences of specialization of the corresponding subgroups, $Ipr_1, \dots, Ipr_w$, to be valid proofs. With each Extra resolution we further associate (finitely many) Generic Collapse Extra resolutions precisely as we did in the sieve procedure.

Having constructed the parameters duo limit groups, and their auxiliary resolutions, we have all the tools that are needed to define the multi-function that is specified in theorem 4.4., i.e., to geometrically eliminate imaginaries.

(1) first, we use the predicate that was constructed in proposition 4.5 to isolate the finitely many equivalence classes that are singled out in theorem 3.1. Note that these (finitely many) equivalence classes are ordered according to the construction of the rigid and solid limit groups, $Ipr_1, \ldots, Ipr_w$, that is used in theorem 3.1. Hence, if the given equivalence relation, $E(p,q)$, is coefficient-free, the predicate that defines these finitely many classes and their order are coefficient-free as well.

(2) Given a collapse uniformization limit group, $ColUnif$, and one of its (finitely many) associated parameters duo limit groups, we look at the equivalence classes that are associated with them. With such an equivalence class, the given finite sets of elements from the ambient collapse uniformization limit group (or alternatively from the parameters duo limit group), that are associated with the edge groups in the graph of groups that is inherited by the collapse extra limit group that is mapped into the collapse uniformization limit group, $ColUnif$, (see part (4) of lemma 3.18 for the elements that are associated with these edge groups) should have only boundedly many values that are associated with each equivalence class, up to conjugation, and right, left and double cosets of the corresponding cyclic groups (the bound is uniform and it does not depend on the equivalence class). Furthermore, there should exist test sequences of the parameters duo limit group that are associated with the equivalence class, and with the specific values of the elements that are associated with the edges in the inherited graph of groups.

    Given the auxiliary resolutions that are associated with the parameters duo limit group, the existence of such test sequences can be expressed using a first order predicate. The parameters that are associated with the equivalence class, are the values of the finite sets of elements of the ambient collapse uniformization limit group, that are associated with the edge groups in the graph of groups that is inherited by the image of the collapse extra limit group from the ambient collapse uniformization limit group, $ColUnif$. The number of these values is uniformly bounded up to the basic definable equivalence relations: conjugation, left, right and double cosets of cyclic subgroups.

    By construction if $E(p,q)$ is coefficient free, so is the predicate that defines the parameters that are associated with the equivalence classes of it.

The parameters that are associated with each equivalence class of $E(p,q)$, according to parts (1) and (2), gives a definable multi-function that is associated with the given definable equivalence relation, $E(p,q)$, with the properties that are listed in theorem 4.4. This function is coefficient-free if $E(p,q)$ is coefficient-free.

□

By the results of [Se8], all the constructions that were associated with a (definable) equivalence relation over a free group can be associated with a definable equivalence relation over a (non-abelian) torsion-free hyperbolic group. Hence, torsion-free hyperbolic groups admit the same type of geometric elimination of imaginaries as a non-abelian free group.

Theorem 4.6. *Let Γ be a non-elementary, torsion-free hyperbolic group, and let $E(p,q)$ be a definable equivalence relation over Γ. The conclusion of theorem 4.4*

holds for $E(p,q)$. If we add sorts for the imaginaries that are presented in section 2: conjugation, left and right cosets of cyclic groups, and double cosets of cyclic groups (all in Γ), then $E(p,q)$ is geometrically eliminated.

Proof: By [Se8] the description of a definable set over a hyperbolic group is similar to the one over a free group. In [Se8], the analysis of solutions to systems of equations, the construction of formal solutions, the analysis of parametric equations, and in particular the uniform bounds on the number of rigid and strictly solid families of solutions that are associated with a given value of the defining parameters, are generalized to non-elementary, torsion-free hyperbolic groups. Furthermore, in [Se8] the sieve procedure is generalized to torsion-free hyperbolic groups, as well as the the analysis and the description of definable sets. Duo limit groups are defined and constructed over torsion-free hyperbolic groups precisely as over free groups, and so are the Diophantine envelope and the duo envelope (see section 1).

Finally, theorem 1.3 in [Se3] that guarantees that given a f.g. group, and a sequence of homomorphisms from G into a free group, there exists an integer s, and a subsequence of the given homomorphisms that converges into a free action of some limit quotient of G on some R^s-tree, remains valid over torsion-free hyperbolic groups. This is the theorem that is used to prove the termination of the iterative procedure for separation of variables, i.e., the iterative procedure for the construction of uniformization limit groups.

Therefore, the procedure for separation of variables that was presented in the previous section generalizes to torsion-free hyperbolic groups, and so is the modification of the sieve procedure that allows us to run the sieve procedure while preserving the separation of variables, that was described in this section. Hence, with a given equivalence relation, $E(p,q)$, over a torsion-free hyperbolic group, one can associate a finite collection of uniformization and Collapse uniformization limit groups, precisely as over free groups. By the argument that was used to prove theorem 4.4 (that remains valid over torsion-free hyperbolic groups), this collection of Collapse uniformization limit groups, enables one to geometric elimination of imaginaries (over torsion-free hyperbolic groups), when we add sorts for the 3 basic families of imaginaries (conjugation, left and right cosets, and double cosets of cyclic groups).

□

References

[Be-Fe1] M. Bestvina and M. Feighn, *Bounding the complexity of simplicial group actions*, Inventiones Mathematicae **103** (1991), 449-469.

[Be-Fe2] ———, *Negligible sets*, preprint.

[Pi1] A. Pillay, *An introduction to stability theory*, Oxford university press, 1983.

[Pi2] ———, *Geometric stability theory*, Oxford university press, 1996.

[Se1] Z. Sela, *Diophantine geometry over groups I: Makanin-Razborov diagrams*, Publication Math. de l'IHES **93** (2001), 31-105.

[Se2] ———, *Diophantine geometry over groups II: Completions, closures and formal solutions*, Israel jour. of Mathematics **134** (2003), 173-254.

[Se3] ———, *Diophantine geometry over groups III: Rigid and solid solutions*, Israel jour. of Mathematics **147** (2005), 1-73.

[Se4] ———, *Diophantine geometry over groups IV: An iterative procedure for validation of a sentence*, Israel jour. of Mathematics **143** (2004), 1-130.

[Se5] ———, *Diophantine geometry over groups V_1: Quantifier elimination I*, Israel jour. of Mathematics **150** (2005), 1-197.

[Se6] ———, *Diophantine geometry over groups V_2: Quantifier elimination II*, GAFA **16** (2006), 537-706.

[Se7] ———, *Diophantine geometry over groups VI: The elementary theory of a free group*, GAFA **16** (2006), 707-730.

[Se8] ———, *Diophantine geometry over groups VII: The elementary theory of a hyperbolic group*, Proceedings of the LMS **99** (2009), 217-273.

[Se9] ———, *Diophantine geometry over groups VIII: Stability*, preprint.